NON-EXISTENCE AND FINITENESS RESULTS FOR TEICHMÜLLER CURVES IN PRYM LOCI

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Abstract. The minimal stratum in Prym loci have been the first source of infinitely many primitive, but not algebraically primitive Teichmüller curves. We show that the stratum Prym(2,1,1) contains no such Teichmüller curve and the stratum Prym(2,2) at most 92 such Teichmüller curves.

This complements the recent progress establishing general – but non-effective – methods to prove finiteness results for Teichmüller curves and serves as proof of concept how to use the torsion condition in the non-algebraically primitive case.

Contents

1. Introduction

Teichmüller curves are isometrically immersed algebraic curves $C = \mathbb{H}/\Gamma \to \mathcal{M}_g$ in the moduli space of genus g Riemann surfaces. These arise as $\mathrm{SL}_2(\mathbb{R})$-orbits of special flat surfaces (X, ω) or half-translation surfaces (X, q) that are called Veech surfaces. By a canonical double covering construction half-translation surfaces can be reduced to flat surfaces, the classification problem for Teichmüller curves is primarily focused on those generated by flat surfaces. We will exclusively deal with this case in the sequel.

A Veech surface is called *primitive*, if it is does not arise from a flat Veech surface of lower genus via a covering construction. The *trace field* $K = \mathbb{Q}(\mathrm{Tr}(\gamma) : \gamma \in \Gamma)$ is a useful invariant of a Veech surface. If $[K : \mathbb{Q}] = g$, then the Veech surface is called *algebraically primitive*. This notion implies that the Veech surface is primitive, but the converse does not hold, examples being given by Teichmüller curves in Prym

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loci that we define below. Teichmüller curves are called (algebraically) primitive, if the generating Veech surfaces have this property.

The classification problem for Teichmüller curves can be subdivided into the classification of primitive Teichmüller curves and the classification of covering constructions. Current progress towards the classification of Teichmüller curves consists of results in three different flavors.

First, for low genus and Veech surfaces with a single zero, there are complete classification results, using the geometry of prototypes. This applies to the classification of primitive Teichmüller curves in genus two ([McM05]) and in the Prym loci ([LN14]), see Section 2.1 for their definition.

Second, there are (in principle effective) finiteness results for algebraically primitive Teichmüller curves in the hyperelliptic components of $\Omega\mathcal{M}_g(g-1,g-1)$ ([Möl08]) and in genus three ([BHM16]). However, only for $\Omega\mathcal{M}_2(1,1)$ there is a complete classification ([McM06b]). In the other cases, the theoretical bounds given by the proofs cannot be directly translated into feasible algorithms and should be combined with techniques recently developed.

The third group of results consists of non-effective finiteness theorems based on equidistribution results of Eskin, Mirzakhani and Mohammadi ([EM13] and [EMM15]). These methods apply e.g. to the algebraically primitive case in prime genus with a single zero ([MW15]). The most general result in this direction was proven recently by Eskin, Filip and Wright ([EFW]) who show in every genus the finiteness of the number of Teichmüller curves with trace field of degree greater than two and more generally finiteness outside special affine invariant submanifolds. This is complemented by the finiteness of Teichmüller curves in non-arithmetic rank-one orbit closures proven in [LNW15]. Yet another route towards non-effective finiteness results is taken by Hamenstädt ([Ham17]).

In this paper we examine two loci in genus three that could, in the light of the results above, potentially contain infinitely many primitive (but not algebraically primitive) Teichmüller curves. The loci under consideration belong to strata with several zeros, so that the torsion condition from [Möl06a] gives additional constraints on the existence of such Teichmüller curves. The main purpose here is to show how to use this torsion condition in the non-algebraically primitive case to prove effective finiteness results.

Theorem 1.1. *There is at most* 92 *primitive Teichmüller curves lying inside the Prym locus* $\mathrm{Prym}_3(2,2)$ *in* $\Omega\mathcal{M}_3$. *The possible examples have trace fields* $\mathbb{Q}[\sqrt{2}]$, $\mathbb{Q}[\sqrt{3}]$ *or* $\mathbb{Q}[\sqrt{33}]$.

In a stratum with one more zero, we push the argument to a complete classification.

Theorem 1.2. *The Prym locus* $\mathrm{Prym}_3(2,1,1)$ *in* $\Omega\mathcal{M}_3$ *does not contain any primitive Teichmüller curve.*

We briefly sketch the proof of the main theorems. It involves degeneration of surfaces to the boundary of the moduli space, and then an analysis of the stable differentials. We show that the number of possible parameters of a periodic direction is finite. This last step towards finiteness is tackled using the Thurston-Veech's construction.

In each step we complement the theoretical argument (implying the finiteness claim) by a way to implement this step in practice.

We suspect that the stratum $\mathrm{Prym}_3(2,2)$ does not contain any primitive Teichmüller curve either. One way to prove this would be to build the flat surfaces given by Thurston-Veech construction with the explicit tiles given in Section 8 and tediously check that in a transverse direction the ratios of moduli are not commensurable. This last check has been done successfully for one of the eight topological models.

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2. Prym loci and eigenforms

2.1. **Prym loci.** In this context, a double covering $\pi : X \to Y$ of two Riemann surfaces is called a *Prym covering*, if $g(X) - g(Y) = 2$. We refer to the involution ρ of X with $Y = X/\langle\rho\rangle$ as the *Prym involution.* This terminology is taken from [McM06a] and differs from the classical terminology, where the polarization on the Prym variety $\mathrm{Prym}(X, \rho) = \mathrm{Jac}\,(X)/\pi^*\mathrm{Jac}\,(Y)$ induced from the principal polarization on $\mathrm{Jac}\,(X)$ had to be a multiple of a principal polarization, but otherwise no genus restriction was imposed.

The *Prym locus* $\mathrm{Prym}_g(\kappa)$ is defined for any partition κ of $2g-2$ to be the set of flat surfaces (X, ω) such that $g(X) = g$, such that X admits a Prym involution and such that the zeros of ω are of type κ. For any (X, ω) in a Prym locus, the quadratic differential $q = \omega^2$ is ρ-invariant, hence a pull-back of from Y. Consequently, there is an $\mathrm{SL}_2(\mathbb{R})$-invariant isomorphism between the Prym loci and strata of quadratic differentials on Y.

Note that a zero of odd order is never fixed by the Prym involution, but zeros of even order may be fixed or interchanged. Consequently, there are two strata with $\kappa = (2, 2)$. If the two zeros are fixed, the Prym locus is isomorphic to $Q(1, 1, -1, -1)$ and part of the hyperelliptic component of $\Omega\mathcal{M}_3(2, 2)^{\mathrm{hyp}}$. If the two zeros are interchanged, the stratum Prym locus is isomorphic to $Q(4, -1^4)$ and part of $\Omega\mathcal{M}_3(2, 2)^{\mathrm{odd}}$. Consequently, we denote these two components by $\mathrm{Prym}_3(2, 2)^{\mathrm{hyp}}$ and $\mathrm{Prym}_3(2, 2)^{\mathrm{odd}}$ respectively. Note that these two components are not of the same dimension.

We give the full list of Prym loci in the case $g = 3$ we are most interested in. These are $\mathrm{Prym}_3(4)$ isomorphic to $Q(3, -1^3)$, the loci $\mathrm{Prym}_3(2, 2)^{\mathrm{hyp}}$ and $\mathrm{Prym}_3(2, 2)^{\mathrm{odd}}$ mentioned above, the locus $\mathrm{Prym}_3(2, 1, 1)$ isomorphic to $Q(2, 1, -1^3)$ and the locus $\mathrm{Prym}_3(1, 1, 1, 1)$ isomorphic to $Q(2^2, -1^4)$. These five Prym loci in genus three are in fact connected, see [LN16a] for the classification of connected components of Prym loci.

2.2. **Prym eigenform loci.** The main observation of [McM06a] was that the *Prym eigenform loci*, defined as intersection with the real multiplication locus

$$\Omega\mathcal{E}_D(\kappa) \;=\; \{(X, \omega) \in \mathrm{Prym}_g(\kappa) \;:\; \mathrm{Prym}(X, \rho)\,\text{has real multiplication by}\,\mathfrak{o}_D\}$$

are $\mathrm{SL}_2(\mathbb{R})$-invariant for any discriminant D and a partition κ of $2g-2$ with $g \in \{2,3,4,5\}$.

Teichmüller curves in Prym loci are necessarily contained in the Prym eigenform loci by [Möl06b]. We summarize when they coincide and the current state of knowledge about these interesting curves.

The intersection $\mathrm{Prym}_g(2g-2)$ with the minimal stratum consists of a union of Teichmüller curves for $g = 2,3,4$ and this intersection is empty otherwise ([McM06a]). The connected components have been classified for $g = 2$ in [McM05] and in [LN14; LN17b] for $g = 3,4$. The topology of these curves is quite well-known. The Euler characteristic has been determined in [Bai07] for $g = 2$ and in [LN16b] for $g = 3,4$. The number of cusps is calculated for $g = 2$ in [McM05] and in [LN14] for $g = 3,4$. The elliptic elements are known for $g = 2$ by work of Mukamel ([Muk14]) and for $g = 3,4$ by work of Torres and Zachhuber ([TZ15], [TZ16]).

The eigenform loci $\Omega\mathcal{E}_D(2,2)$ are also two-dimensional, hence union of Teichmüller curves. Almost all of the obvious topological questions on these curves are currently open.

The situation is rather different in the cases when the dimension of the eigenform locus in $\mathrm{Prym}_g(\kappa)$ is three or larger. The components of the loci $\mathrm{Prym}_3(2,2)^{\mathrm{odd}}$ and $\mathrm{Prym}_3(2,1,1)$, which are of particular interest in this paper, have been classified in [LN17a]. The same questions (connected components, classification of Teichmüller curves) naturally also arise in $\mathrm{Prym}_3(1,1,1,1)$ and also in genus 4 and 5. They are currently open, mainly since the number of configurations of cylinders grows drastically with the dimension of the loci.

Note that all surfaces in the eigenform loci that are given by the Thurston-Veech construction (see [McM06a, Section 4] for a concise survey) are completely algebraically periodic in the sense of [CS08]. Hence all these surfaces have zero flux and we cannot use the flux to rule out the existence of Teichmüller curves. Results on the more interesting property of complete periodicity in Prym loci can be found in [LN16a]. However, this property is not relevant here either.

3. Suitable degenerations

It has been a recurrent theme in [McM06b], [Möl08] , [BM12] that the constraints imposed by the torsion condition of [Möl06a] are best expressible on a the stable curves over cusps of Teichmüller curves. This applies in particular, if these stable curves are irreducible and rational curves with nodes.

For primitive, but not algebraically primitive Teichmüller curves stable curves that are non-rational may appear at cusps. However, for the Prym loci under consideration we can easily find suitable cusps.

The torsion condition translates into an equation in roots of unity for any suitable cusp. In this section we derive this equation in roots of unity both for $\mathrm{Prym}_3(2,2)$ and $\mathrm{Prym}_3(2,1,1)$.

3.1. **Stable curves, stable differentials and the torsion condition.** Let $\overline{\mathcal{M}}_g$ denote the Deligne-Mumford compactification of the moduli space of curves. A point in the boundary of $\overline{\mathcal{M}}_g$ is a *stable curve* Y, i.e. a smooth curve $\widetilde{Y}$ with certain pairs of points (x_i, y_i) identified to form nodes such that the automorphism group of

Y has finite order. A *stable form* η on Y is a holomorphic 1-form on $\widetilde{Y} \setminus \bigcup_{i=1}^n (x_i, y_i)$ with at worst simple poles of opposite residues at each pair of points (x_i, y_i).

Given translation surface (X, ω) and a completely periodic direction Θ, one can use the diagonal matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & s \end{smallmatrix}\right)$ to push (X, ω) to the boundary of $\mathcal{M}_g$. The limit point corresponds to a cusp of the Teichmüller curve $\overline{C} \setminus C$. Topologically the stable surface is obtained by collapsing the core curves of the cylinders to points, i.e. each cylinder gives rise to a node and the irreducible components of the surface are obtained by cutting along the core curves.

This description also makes evident that the projective tuple of residues of η equals the tuple of widths of the cylinders.

We call a direction Θ on a Veech surface and the corresponding cusp of the Teichmüller curve *irreducible* if the stable curve is irreducible. We call Θ and the cusp *suitable* if the stable curve is moreover a rational curve.

We recall here the torsion condition that we want to exploit in our setting. Suppose (X, ω) is a Veech surface and that ω has zeros $z_1, \ldots, z_m$. Let $\mathrm{Jac}\,(X)^- = \mathrm{Jac}\,(X)/\mathrm{Jac}\,(X/\rho)$ be the ρ-anti-invariant part of the Jacobian of X.

Theorem 3.1 ([Möl06a])**.** *For any i, j the divisor $[z_i - z_j]$ has finite order in* $\mathrm{Jac}\,(X)^-$.

3.2. **Suitable cusps in the case** $\mathrm{Prym}_3(2,1,1)$**.** We first show that directions of a saddle connection joining the double zero to a simple zero are suitable for primitive Teichmüller curves in this locus.

Lemma 3.2. *Let $(X, \omega) \in \mathrm{Prym}_3(2,1,1)$ be a Veech surface generating a primitive Teichmüller curve. If Θ is the direction of a saddle connection joining the double zero to a simple zero, then the stable curve is an irreducible rational curve of geometric genus zero, i.e. the direction is suitable.*

The cylinder decomposition in the direction Θ consists of three cylinders, one fixed by ρ and a pair that is exchanged by ρ.

Convention 1. We will always label the cylinders of a suitable direction such that C_2 is fixed by the Prym involution ρ, while the cylinders C_1 and C_3 are permuted. The widths and heights of the cylinders are denoted by $w_i = w(C_i)$ and $h_i = h(C_i)$.

Proof of Lemma 3.2. Let Θ be a direction Θ on (X, ω) with a saddle connection joining the double zero to a simple zero. Let Y be the stable surface obtained by degenerating using the geodesic flow in the direction Θ. One component of the stable curve thus contains the double zero, a simple zero. This component is fixed by the Prym involution since the double zero is fixed and hence it also contains the other simple zero. Consequently, Y is an irreducible curve. Since $g(X) = 3$, the curve Y has at most three cusps and the direction Θ at most three cylinders. Since (X, ω) is primitive the widths of the core curve of the cylinders have $\mathbb{Q}$-rank at least two. If two cylinders were fixed, we would obtain a contradiction to the double zero being fixed. Hence one cylinder is not fixed and this direction has precisely three cylinders.

This determines the genus of the normalization of Y. It is isomorphic to a rational curve $\mathbb{P}^1$ with three pairs of points that are identified. □

Next, we express the stable form of a suitable direction algebraically. We use z as a coordinate on $\mathbb{P}^1 \simeq \widetilde{Y}$. The Prym involution ρ is preserved along the degeneration process, and induces an involution, still denoted by ρ, on $\mathbb{P}^1$. We use the triple transitivity of Möbius transformations to make the Prym involution ρ on $\mathbb{P}^1$ to be $z \mapsto -z$. Since the direction Θ decomposes the surface into three cylinders, it turns out that ω_Θ has one double zero, two simple poles and three pairs of poles (with opposite residues). From the action of ρ on cylinders and zeros we deduce that one pair of poles is fixed while the other pairs are exchanged. Hence the pairs of identified points are $(u,-u)$, (x,y) and $(-x,-y)$. The stable differential ω_Θ is therefore given by,

$$\omega_\Theta \;=\; C \cdot \frac{z^2(z^2-1)}{(z^2-x^2)(z^2-y^2)(z^2-u^2)}dz \tag{1}$$

for some nonzero complex scalar C.

Observe that Convention 1 implies that the residue of ω_Θ at x is r_1, and the residue at u is w_2. We can always rescale such that r_1, r_2 form a basis of a real quadratic field K. Stability of ω_Θ implies that the residue around y is $-w_1$, namely

$$\omega_\Theta \;=\; r_1(\frac{1}{z-x} - \frac{1}{z-y})dz + r_2(\frac{1}{z-u} - \frac{1}{z+u})dz - r_1(\frac{1}{z+x} - \frac{1}{z+y})dz. \tag{2}$$

Hence using coefficient comparison between equations (2) and (1), we get the *opposite residue equations*

$$\begin{aligned} 0 \;&=\; r_1 u(y-x) + r_2 xy \\ 0 \;&=\; r_1(yx^2 + (-y^2 + (-u^2+1))x + (u^2-1)y) - r_2 u(x^2+y^2-1)\,, \end{aligned} \tag{3}$$

where the first one is obtained from the constant term and the second one is obtained from the comparison of the z^4-term and z^2-terms. Note that the terms for odd powers of z are automatically zero.

Finally, we express the torsion condition in these coordinates. The universal cover of $\mathrm{Jac}\,(X)^-$ is $\mathbb{C}^2$, generated by ω and ω^σ. The zeros z_i of ω give rise to sections Z_i of the universal family of curves over the Teichmüller curve C. If $z_i - z_j$ is torsion for all fibers over C, then the same holds for the corresponding Jacobian over the cusp. In the case of a suitable degeneration this is the semiabelian variety

$$\mathbb{C}^2/\langle \mathrm{Per}(\gamma_x), \mathrm{Per}(\gamma_u)\rangle \;\cong\; (\mathbb{C}^*)^2,$$

where γ_x and γ_u are the loops around x and u respectively and where Per denotes the vector of periods with respect to ω and ω^σ. The definition of the stable form in (2) implies that the vector space of periods is

$$\mathrm{Per}(\gamma_x) \;=\; \mathbb{Z}\cdot\begin{pmatrix} 2\pi I\cdot r_1 \\ 2\pi I\cdot r_1^\sigma \end{pmatrix}, \quad \mathrm{Per}(\gamma_x) \;=\; \mathbb{Z}\cdot\begin{pmatrix} 2\pi I\cdot r_2 \\ 2\pi I\cdot r_2^\sigma \end{pmatrix}.$$

On the other hand we calculate that the divisor given as the difference between the double zero a simple zero is the vector of relative periods

$$\int_0^1 \omega_\Theta \;=\; r_1 \log\left(\frac{(1-x)(1+y)}{(1+x)(1-y)}\right) + r_2 \log\left(\frac{1-u}{1+u}\right) \tag{4}$$

up to the contribution of closed paths, and similarly for ω_Θ^σ, replacing r_i by r_i^σ.

The torsion condition now amounts to

$$\begin{pmatrix} \int_0^1 \omega_\Theta \\ \int_0^1 \omega_\Theta^\sigma \sigma \end{pmatrix} \;\in\; \mathbb{Q}\cdot\begin{pmatrix} 2\pi I\cdot r_1 \\ 2\pi I\cdot r_1^\sigma \end{pmatrix} + \mathbb{Q}\cdot\begin{pmatrix} 2\pi I\cdot r_2 \\ 2\pi I\cdot r_2^\sigma \end{pmatrix} \tag{5}$$

This is possible only if both logarithms in equation (4) lie in $2\pi I \cdot \mathbb{Q}$. It will be convenient to make the invertible change of variables

$$X = \frac{1-x}{1+x}, \quad Y = \frac{1-y}{1+y}, \quad U = \frac{1-u}{1+u}.$$

With this notation, the torsion condition amounts the existence of two roots of unity ζ_{XY} and ζ_U, such that

$$Y/X = \zeta_{XY} \quad \text{and} \quad U = \zeta_U. \tag{6}$$

We plug (6) solved for Y in the first opposite residue equation (3) and clearing denominators the get the first equation in (7). We perform the same replacement with the second opposite residue equation and subtract a suitable multiple of the first equation to reduce the degree in X from 4 to three. The resulting equations are

$$\begin{aligned} 0 &= r_2(\zeta_U+1)\zeta_{XY}\, X^2 \\ &+ (2r_1(\zeta_{XY}-1)(\zeta_U-1) - r_2(1+\zeta_{XY})(1+\zeta_U))\, X \\ &+ r_2(\zeta_U+1) \qquad\qquad \text{and} \\ 0 &= \zeta_{XY}(16\zeta_U(\zeta_{XY}-1)r_1 - (\zeta_U^2-1)(\zeta_{XY}+1)r_2)\, X^3 \\ &+ (\zeta_U-1)(2(\zeta_U-1)(\zeta_{XY}^2-1)r_1 - (\zeta_U+1)(\zeta_{XY}^2-13\zeta_{XY}+1)r_2)\, X^2 \\ &+ 2((\zeta_{XY}-1)(\zeta_U^2+6\zeta_U+1)r_1 - (\zeta_U^2-1)(\zeta_{XY}+1))X \\ &+ r_2(\zeta_U^2-1). \end{aligned} \tag{7}$$

It turns out that the resultant of the two above equations with respect to X, divided by the factor $256(\zeta_U+1)\zeta_{XY}r_2$, is a square of the following equation

$$\begin{aligned} 0 &= r^2\zeta_{XY}\zeta_U^3 - r^2\zeta_{XY} + (4-r^2)\zeta_{XY}\zeta_U + (r^2-4)\zeta_{XY}\zeta_U^2 \\ &+ (2-r)\zeta_{XY}^2\zeta_U^2 + (r+2)\zeta_U^2 - (r+2)\zeta_{XY}^2\zeta_U + (r-2)\zeta_U. \end{aligned} \tag{8}$$

With the application in the next section in mind, we have group the individual summands as powers of ζ_{XY} and ζ_U. and we have scaled the projective tuple (r_1, r_2) to $r_1 = 1$ and $r_2 = r$ to improve readability.

3.3. **Suitable cusps in the case** $\mathrm{Prym}_3(2,2)$**.** With a proof similar to that of Lemma 3.2 we obtain the following criterion for suitable directions.

Lemma 3.3. *Let* $(X,\omega) \in \mathrm{Prym}_3(2,2)$ *be a Veech surface generating a primitive Teichmüller curve. If* Θ *is the direction of a saddle connection joining the double zeros to a fixed point of the Prym involution, then the stable curve is an irreducible rational curve of geometric genus zero, i.e. the direction is suitable.*

The cylinder decomposition in the direction Θ *consists of three cylinders, one fixed by* ρ *and a pair that is exchanged by* ρ*.*

The saddle connection in the statement of the lemma is joining the two double zeros, with a fixed point at its midpoint.

Proof. In comparison with the proof of Lemma 3.2 we only have to rule out that the direction Θ has two cylinders, both fixed by ρ. Since then all four fixed points are contained in the cylinders this contradicts the hypothesis on the saddle connection. □

Consequently, we may stick to the labelling of cylinders as in Convention 1 here, too. We also continue to use ρ to be $z \mapsto -z$ on the normalization $\widetilde{Y} \cong \mathbb{P}^1$ of any suitable cusp. However, now both zeros are fixed by the Prym involution, so that

$$\omega_\Theta \;=\; C \cdot \frac{(z-1)^2(z+1)^2}{(z^2-x^2)(z^2-y^2)(z^2-u^2)}\, dz.$$

Comparing with

$$\omega_\Theta \;=\; r_1(\frac{1}{z-x} - \frac{1}{z-y})dz + r_2(\frac{1}{z-u} - \frac{1}{z+u})dz - r_1(\frac{1}{z+x} - \frac{1}{z+y})dz.$$

we obtain the *opposite residue equations*

$$\begin{aligned} 0 \;&=\; (x-y)(xyu^2+1)r_1 - u(xy-1)(xy+1)r_2\,, \\ 0 \;&=\; (y-x)(-xy+u^2-2)r_1 - u(x^2+y^2-2)r_2. \end{aligned} \tag{9}$$

The relative period is now given as the integral between the two zeros of ω

$$\int_{-1}^{1} \omega_\Theta \;=\; 2r_1 \log\left(\frac{(1-x)(1+y)}{(1+x)(1-y)}\right) + 2r_2 \log\left(\frac{1-u}{1+u}\right) \tag{10}$$

(up to the contribution of closed paths). The torsion condition implies the existence of two roots of unity ζ_{XY} and ζ_U such that

$$\zeta_U = U \qquad \text{and} \qquad \zeta_{XY} = Y/X, \text{ where } U = \frac{1-u}{1+u},\; X = \frac{1-x}{1+x},\; Y = \frac{1-y}{1+y}.$$

Similarly to the situation in $\mathrm{Prym}_3(2,1,1)$ we plug this (solved for X) into the opposite residue equations and clear denominators. We obtain two expression of degree two in X. A suitable linear combination of the two is $64X$ times the following relation, that we again specialized to $r_1 = 1$, and $r_2 = r$.

$$\begin{aligned} &r(r-1)\zeta_{XY} \;+\; r(r-1)\zeta_{XY}^2\zeta_U^4 \;-\; 2(r^2-1)\zeta_{XY}\zeta_U^2 \;-\; 2(r^2-1)\zeta_{XY}^2\zeta_U^2 \\ &+\; r(r+1)\zeta_{XY}^2 \;+\; r(r+1)\zeta_{XY}\zeta_U^4 \;-\; 2\zeta_{XY}^3\zeta_U^2 \;-\; 2\zeta_U^2 \;=\; 0. \end{aligned} \tag{11}$$

3.4. **Degenerate cases of Equation** (8)**.** We analyze degenerate cases of Equation (8) in order to eliminate them from the discussion in the next section. Solutions with $\zeta_{XY} = 1$ or $\zeta_U = 1$ are of no interest for us since then $x = y$ (respectively, $u = 0$) which would imply that two poles (respectively a pole and a zero) of the stable come together. Idem for $\zeta_U = -1$.

3.5. **Double coverings of** $\mathcal{Q}(1,1,-1^6)$**.** Similarly we discuss the degenerate cases of Equation (11). If $\zeta_U = \pm 1$ then the poles corresponding to $\pm u$ have collided or they have collided with the location $z = \pm 1$ of a zero. If $\zeta_{XY} = 1$ then the poles x and y have collided. We can exclude these cases.

If $(\zeta_{XY}, \zeta_U) = (-1, \pm\imath)$ then Equation (11) holds for any r. In this situation, $x = y^{-1}$ and $u = \pm\imath$. Then the involution $h : z \mapsto 1/z$ fixes all the cylinders and the zeros. It has thus 8 fixed points, i.e. h is a limiting case of the hyperelliptic involution and the quotient belongs to $\mathcal{Q}(1,1,-1^6)$. Since all the cylinders are fixed, hyperelliptic open-up argument [CM12, Proposition 3.4] applies and we conclude that any Teichmüller curve limiting to such a boundary point is a family of hyperelliptic curves. Since the hyperelliptic involution is unique, h and ρ commute, they generate a $(\mathbb{Z}/2)^2$, whose remaining involution we denote by τ. Since

$3 = g(X/\rho)+g(X/\tau)+g(X/h)$ we conclude that X/τ is a genus two curve. If (X,ω) is a generating Veech surface, then $\tau^*\omega = h^*\rho^*\omega = \omega$, hence ω is a pullback from X/τ.

To summarize, boundary points with $(\zeta_{XY}, \zeta_U) = (-1, \pm\imath)$ lie only on Teichmüller curves generated by imprimitive Veech surfaces and need not be considered for the proof of Theorem 1.1.

4. Solving linear relations in roots of unity.

Equations (8) and (11) are instances of what Mann ([Man65]) calls linear relations in roots of unity. More precisely, a *K-relation of length k* is an equation

$$\sum_{i=1}^{k} a_i\zeta_i \;=\; 0$$

where the ζ_i are pairwise different roots of unity and where all a_i lie in the number field K. The relation is called *irreducible*, if $\sum_{i=1}^k b_i\zeta_i = 0$ and $b_i(a_i - b_i) = 0$ for all i implies that $b_i = 0$ for all i or $a_i - b_i = 0$ for all i. Obviously each relation is a sum of irreducible relations, but there may be several ways of writing a relation as sum of irreducible relations.

Mann proved a finiteness statement for $\mathbb{Q}$-relations in roots of unity. This was generalized in [Möl08] to coefficients in a number field of bounded degree. We use here the following version with better bounds of Dvornicich-Zannier on the possible orders of roots of unity. We state the special case of $[K:\mathbb{Q}] = 2$.

Theorem 4.1 ([DZ00, Theorem 1])**.** *Suppose that $\sum_{i=1}^k a_i\zeta_i = 0$ is an irreducible K-relation. Then there exists a primitive N-th root unity ζ_N, some root of unity ξ and exponents $b_i \in \mathbb{Z}$ such that $\zeta_i = \xi\zeta_N^{b_i}$, where N is bounded as follows.*

The exponents of primes dividing N are bounded by the condition that if $p^{\alpha+1}|N$ for some prime $p \in \mathbb{Z}$ then $p^\alpha \,|\, 2d$ where $d := [K:\mathbb{Q}]$. Moreover the size of primes dividing N is bounded by

$$\sum_{p|N}\left(\frac{p-1}{(p-1,d)} - 1\right) \le k-2\,.$$

The purpose of this section is to provide algorithms with feasible bounds to find all solutions to Equations (8) and (11) that are relevant from the point of view of cusps of Teichmüller curves. We denote $\mathcal{N}(k)$ the set of orders that may appear for a K-relation of length k according to Theorem 4.1. Note that $\mathcal{N}(k)$ is monotone in k, i.e. if $\ell \ge k$ then $\mathcal{N}(\ell) \subset \mathcal{N}(k)$.

4.1. **The K-relation for** $\mathrm{Prym}_3(2,1,1)$**.** We will prove

Theorem 4.2. *There are at most finitely many solutions of Equation* (8)

$$\begin{array}{lllllllll} r^2\zeta_{XY}\zeta_U^3 & - & r^2\zeta_{XY} & + & (4-r^2)\zeta_{XY}\zeta_U & + & (r^2-4)\zeta_{XY}\zeta_U^2 & + & \\ (2-r)\zeta_{XY}^2\zeta_U^2 & + & (r+2)\zeta_U^2 & - & (r+2)\zeta_{XY}^2\zeta_U & + & (r-2)\zeta_U & & = 0. \end{array}$$

where r is real, $[\mathbb{Q}(r):\mathbb{Q}] = 2$ and where ζ_{XY}, ζ_U are roots of unity with $\zeta_{XY} \neq 1$ and $\zeta_U \notin \{\pm 1\}$. All the solutions are given by N-th roots of unity for some $N \in \mathcal{N}(8)$.

N	e_{XY}	e_U	r
6	1	1	$\frac{3+\sqrt{33}}{6}$
6	3	1	$\frac{2\sqrt{6}}{3}$
6	3	2	$2\sqrt{2}$
6	5	1	$\frac{-3+\sqrt{33}}{6}$
12	6	1	$-2+2\sqrt{3}$
12	6	5	$2+2\sqrt{3}$
24	3	4	$\frac{2\sqrt{3}}{3}$
24	15	4	$\frac{2\sqrt{3}}{3}$

More precisely, there are only 16 *such solutions with* $N_{\mathbb{Q}}^K(r) < 0$. *They are given in the following table but only one representative from each pair* $(\zeta_{XY}, \zeta_U, r) = (\zeta_N^{e_{XY}}, \zeta_N^{e_U}, r)$ *and* $(\zeta_{XY}, \zeta_U, r) = (\zeta_N^{N-e_{XY}}, \zeta_N^{N-e_U}, r)$.

The reason for restricting to the solutions with $N_{\mathbb{Q}}^K(r) < 0$ will become clear in Section 5.1. Note that since r_2 is real, the solutions come in complex conjugate pairs.

The idea of proof is to combine Theorem 4.1 with the fact that the roots of unity appearing in equation (8) are not arbitrary but rather products of only two roots of unity ζ_{XY} and ζ_U. More precisely we will apply Theorem 4.1 to all possible ways of writing (8) as sums of irreducible K-relations. The following condition is helpful. Suppose an irreducible relation contained in (8) contains at least three terms $a_j\zeta_{XY}^{\alpha_j}\zeta_U^{\beta_j}$ for $j \in J$ with $|J| \geq 3$. We call such an irreducible relation *good*, if there are three indices $i, j, k \in J$, such that

$$d_{ijk} := \det\begin{pmatrix} \alpha_i - \alpha_j & \alpha_i - \alpha_k \\ \beta_i - \beta_j & \beta_i - \beta_k \end{pmatrix}$$

is non-zero.

Lemma 4.3. *Suppose that an irreducible relation* $\sum_{j\in J} a_j\zeta_{XY}^{\alpha_j}\zeta_U^{\beta_j} = 0$ *contained in Equation* (8) *is good and let*

$$d = \gcd_{i,j,k\in J}\{d_{ijk}\}.$$

Then the order of both ζ_{XY} *and* ζ_U *is an element of* $d\mathcal{N}(|J|)$.

Proof. By Theorem 4.1 we know that there exists an N-th root of unity ζ_N with N in $\mathcal{N}(|J|)$ and some root of unity ξ such that $\zeta_{XY}^{\alpha_j}\zeta_U^{\beta_j} = \xi\zeta_N^{a(j)}$ for some integer $a(j)$. For any good triple of indices (i, j, k) we can solve this for ζ_{XY} and ζ_U being a root of unity in $d_{ijk}\mathcal{N}(|J|)$. Considering this condition jointly for all such triples we conclude that they are in fact roots of unity in the set $d\mathcal{N}(|J|)$. □

We will distinguish two cases, K-relations of length two and longer K-relations. K-relations of length two are never good, but they impose other strong constraints and we deal with them separately. Labelling of the terms is in the order they appear in Equation (8).

Lemma 4.4. *If a solution to the Equation* (8) *has a sub-*K*-relation of length two, then at least one of the following statements holds true.*

i) The relation consists of terms (12) and $\zeta_U^3 = 1$.
ii) The relation consists of terms (58) and $\zeta_U = \zeta_{XY}^{-2}$.
iii) The relation consists of terms (67) and $\zeta_U = \zeta_{XY}^{2}$.

Moreover, in this case all the solutions of Equation (8) *are of the form* $(\zeta_{XY}, \zeta_U) = (\zeta_N^{e_{XY}}, \zeta_N^{e_U})$ *for some* $N \in \mathcal{N}(5)$.

Proof. If an irreducible relation contained in (8) has precisely two terms

$$a_1\zeta_{XY}^{\alpha_1}\zeta_U^{\beta_1} + a_2\zeta_{XY}^{\alpha_2}\zeta_U^{\beta_2} \;=\; 0, \quad \text{then} \quad -a_1/a_2 \;=\; \zeta_{XY}^{\alpha_2-\alpha_1}\zeta_U^{\beta_2-\beta_1} \in \{\pm 1\} \qquad (12)$$

since these are the only real roots of unity.

Suppose that the expressions for a_1 and a_2 as they appear in (8) are not equal up to sign. If the a_i are both linear in r this gives a solution $r \in \mathbb{Q}$, contradiction. We run into the same contradiction for $a_1 = \pm(r^2-4)$ and a_2 linear (and vice versa), since the linear terms that appear divide this a_1. For $a_1 = \pm r^2$ and a_2 linear (and vice versa), the solution is either non-real or in $\mathbb{Q}$, contradiction. The case of both a_i being quadratic in r is possible (given that $\zeta_U \neq \pm 1$) only if the relation is (13) or (24). In either case we deduce $\zeta_U^2 = -1$ and $r = \sqrt{2}$. Then we can solve (8) for ζ_{XY}, in fact $\zeta_{XY}^2 = \frac{1}{3}(-1 \pm 2i\sqrt{2})$, which is not solvable for a root of unity.

If the expressions for a_1 and a_2 as they appear in (8) agree up to sign, we are in one of the cases listed, since the relation (34) implies $\zeta_U = \pm 1$. We discuss the resulting equations separately.

In case i) we obtain after taking resultant with $\zeta_U^3 - 1 = 0$ and dividing by $\zeta_{XY}^2 - 1$the relation

$$(r^2+12)\zeta_{XY}^4 + (12r^2-48)\zeta_{XY}^3 + (3r^4-26r^2+72)\zeta_{XY}^2 + (12r^2-48)\zeta_{XY} + r^2 + 12 = 0.$$

In case ii) we obtain after substituting and dividing by $\zeta_{XY}^2 - 1$

$$r^2\zeta_{XY}^4 + (r+2)\zeta_{XY}^3 + (2r^2-4)\zeta_{XY}^2 + (r+2)\zeta_{XY} + r^2 = 0.$$

The relation stemming from the last case iii) is the same as in the previous case after replacing r by $-r$.

If none of the coefficients is zero in any of the two preceding displayed equations, these are either irreducible (hence with a solution for ζ_{XY} in for $N \in \mathcal{N}(5)$) or reducible with a partition in two plus three terms, hence with a solution for ζ_{XY} in for $N \in \mathcal{N}(3)$. Since ζ_U is determined, being a power of ζ_{XY}, or a third root of unity, the claim follows in this case.

None of the coefficients of the first equation is annihilated by a real quadratic irrational number and $r = \frac{\sqrt{2}}{2}$ is the only interesting case for the second equation. Some of the solutions lie on the unit circle, but a loop over the finitely many roots of unity of degree 16 over $\mathbb{Q}$ shows that these solutions are not roots of unity. □

Proof of Theorem 4.2, finiteness statement. The only triples that are not good are (labeling of the terms as they appear in Equation (8)) $(4,5,6)$, $(3,7,8)$ all the triples in $\{1,2,3,4\}$. By the preceding lemma we only need to deal with atom-free partitions of (8) without a part of length two since the number of possibilities for $N \in \mathcal{N}(5)$ is finite. The only possibilities are $(5,3)$, $(4,4)$ and (8). The relation of length 8 is obviously good. By the preceding statement none of the relations of length 5 is bad. If the relation of length 4 is the bad one, consisting of (1234), then the complementary relation is good. Consequently, all these situations are good and Lemma 4.3 gives the finiteness statement. □

4.2. **Implementation for the case** $\mathrm{Prym}_3(2,1,1)$**.** The goal is to prove the second part of Theorem 4.2, namely to show that all the solutions are given by the table in Theorem 4.2. The first step is to reduce the discussion to the partition (8) into just one irreducible relation, since we have to treat this case anyway, and the second step is to feasibly implement this case.

We start with the second step and deal with **Case** (8). We have $\mathcal{N}(8) = 2^3 \cdot 3 \cdot \{17, 5\cdot 13, 7\cdot 11, 5\cdot 11, 5\cdot 7\}$. For each order N in that list and for each pair of exponents (e_X, e_U) we have to try whether for $\zeta_{XY} = \zeta_N^{e_X}$ and $\zeta_U = \zeta_N^{e_U}$ the quadratic equation (in r) with coefficients in that cyclotomic field has a solution in a quadratic number field.

Equation (8) can be rewritten as $ar^2 + br + c = 0$, where

$$\begin{cases} a &= \zeta_N^{e_X}(\zeta_N^{e_U}-1)(\zeta_N^{e_U}+1)^2 \\ b &= -\zeta_N^{e_U}(\zeta_N^{e_U}+1)(\zeta_N^{e_X}-1)(\zeta_N^{e_X}+1) \\ c &= 2\zeta_N^{e_U}(\zeta_N^{e_U}-1)(\zeta_N^{e_X}-1)^2 \end{cases}$$

Observe that $a \neq 0$ since $\zeta_U \neq \pm 1$. We can solve the above quadratic equation as follows. We let $\Delta = \sqrt{b^2 - 4ac}$.

i) If $b/a \in \mathbb{Q}$ and $c/a \in \mathbb{Q}$ then $[\mathbb{Q}(r):\mathbb{Q}] \leq 2$ and we record the value, if the degree is two.
ii) Otherwise, if $b/a \in \mathbb{Q}$ but $\sqrt{\Delta}/a \notin \mathbb{Q}(\zeta_N)$, then $\sqrt{\Delta}/a$ belongs to a quadratic extension of $\mathbb{Q}$, linearly disjoint from $\mathbb{Q}(\zeta_N)$. But then $\Delta/a^2 \in \mathbb{Q}$, hence $c/a \in \mathbb{Q}$, contradiction. If $b/a \notin \mathbb{Q}$ and $\sqrt{\Delta}/a \notin \mathbb{Q}(\zeta_N)$, then $2r = b/a \pm \sqrt{\Delta}/a$ cannot have degree two over $\mathbb{Q}$.
iii) If $\sqrt{\Delta}/a \in \mathbb{Q}(\zeta_N)$, compute r as element of $\mathbb{Q}(\zeta_N)$ and record the value, if the degree over $\mathbb{Q}$ is two.

For practical purposes the loop over N^2 cases can be simplified significantly. First, we may suppose that $\gcd(e_x, e_U, N) = 1$, since otherwise the case has been treated before, for a divisor of N. Suppose first that we are in the 'rational' case i). Then for any $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ we need to find common solutions of $ab^\sigma - a^\sigma b = 0$ and $ac^\sigma - a^\sigma c = 0$. Taking resultants, this reduces to a loop over just one variable.

Suppose now that $r \in \mathbb{Q}(\zeta_N)$ as in case iii). Then $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ acts on the set of solutions of (8) without preserving the condition $N(r) < 0$. Using this action we may suppose that $e_X | N$, thus reducing the loop to N times the number of divisors of N cases.

Testing if Δ has a square root in $\mathbb{Q}(\zeta_N)$ in step iii) is a time-consuming operation. In practice it works much better to assume that r_2 satisfies a quadratic equation $r_2^2 + \beta r_2 + \gamma = 0$, to take the resultant with (8) and solve the resulting system for rational number β, γ. If this system does not determine β, γ up to finitely many choice, we can still fall back on the original test.

We now discuss step one. Suppose that one of the relations, say of length k, is good and $d = 1$. Then we find all solutions by a loop over all $N \in \mathcal{N}(k)$, all e_X, e_U and $\zeta_{XY} = \zeta_N^{e_X}$ and $\zeta_U = \zeta_N^{e_U}$ and we need to check if Equation (8) holds for these values form some r, quadratic over $\mathbb{Q}$. Since $\mathcal{N}(\cdot)$ is monotone in the argument, these cases have already been checked. Also, by Lemma 4.4 all the partition with a sub-K-relation of length two have already been checked.

Case (3,5). For every such partition, one of the two relations has $d = 1$, so all the cases have been dealt with.
Case (4,4). The only 4-tuples for which $d \neq 1$ are listed below, with their complementary relation:

	relation	complementary relation	values of d
(1)	$(1, 3, 5, 6)$	$(2, 4, 7, 8)$	2 resp. 2
(2)	$(1, 3, 7, 8)$	$(2, 4, 5, 6)$	2 resp. 2
(3)	$(1, 2, 3, 4)$	$(5, 6, 7, 8)$	0 resp. 2

In each case we consider the resultant of the two equations with respect to ζ_{XY}. In the first two cases this factors completely and we obtain, respectively,

$$16(r-2)(r-1)(r+1)(r+2)(\zeta_U-1)^2(\zeta_U+1)^2\zeta_U^4 \;=\; 0$$

and

$$-(r-2)(r+2)\zeta_U^2(\zeta_U-1)^4(\zeta_U+1)^4r^4 \;=\; 0.$$

Since $\zeta_U \neq \pm 1$ and $r \notin \mathbb{Q}$ there is no solution.

Finally, for the pair of relations in (3) the resultant with respect to ζ_{XY} gives:

$$\zeta_U(\zeta_U-1)^2 \cdot (r(\zeta_U+1)+2\zeta_U-2) \cdot \left(r^2(\zeta_U^2+2\zeta_U+1)-4\zeta_U\right)^2 \;=\; 0.$$

If $r(\zeta_U + 1) + 2\zeta_U - 2 = 0$ then the equation given by terms $(5, 6, 7, 8)$ becomes $\zeta_U(\zeta_U - 1)(\zeta_U + 1)\zeta_{XY}^2 = 0$ leading to a contradiction. The remaining possibility is $r^2(\zeta_U^2 + 2\zeta_U + 1) - 4\zeta_U = 0$. This is a length 3 irreducible relation:

$$r^2\zeta_U^2 + (2r^2 - 4)\zeta_U + r^2 = 0.$$

If $r \neq \sqrt{2}$ the solutions for ζ_U correspond to N-the roots of unity for $N \in \mathcal{N}(3)$. Now using the resultant with respect to ζ_U we find (after dividing by rational factors) a cubic equation for ζ_{XY} which is irreducible unless $r = \sqrt{4/3}$ and ζ_{XY} is an 8-the root of unity. In any case, such a solution has appeared while searching the irreducible case of Equation (8).

If $r = \sqrt{2}$ then the same resultant shows that $\zeta_{XY}^2 = \frac{-3\pm\sqrt{-7}}{4}$. This number lies on the unit circle, but this is not a root of unity.

4.3. **The K-relation for** $\text{Prym}_3(2, 2)$**.** We now discus the relation in roots of unity in the other stratum.

Theorem 4.5. *There are at most finitely many solutions of Equation* (11)

$$\begin{aligned} &r(r-1)\zeta_{XY} \;+\; r(r-1)\zeta_{XY}^2\zeta_U^4 \;-\; 2(r^2-1)\zeta_{XY}\zeta_U^2 \;-\; 2(r^2-1)\zeta_{XY}^2\zeta_U^2 \\ &+\; r(r+1)\zeta_{XY}^2 \;+\; r(r+1)\zeta_{XY}\zeta_U^4 \;-\; 2\zeta_{XY}^3\zeta_U^2 \;-\; 2\zeta_U^2 = 0. \end{aligned}$$

where r is real, $[\mathbb{Q}(r) : \mathbb{Q}] = 2$ and where ζ_{XY}, ζ_U are roots of unity with $\zeta_{XY} \neq 1$, $\zeta_U \notin \{\pm 1\}$ and $(\zeta_{XY}, \zeta_U) \neq (-1, \pm\imath)$. All the solutions are given by N-th roots of unity for some $N \in \mathcal{N}(8) \cup 2 \cdot \mathcal{N}(4) \cup 4 \cdot \mathcal{N}(4)$.

More precisely, there are only 32 such solutions with $N_{\mathbb{Q}}^K(r) < 0$. They are given in the following table but only one representative from each pair $(\zeta_{XY}, \zeta_U, r) = (\zeta_N^{e_{XY}}, \zeta_N^{e_U}, r)$ and $(\zeta_{XY}, \zeta_U, r) = (\zeta_N^{N-e_{XY}}, \zeta_N^{N-e_U}, r)$.

N	e_{XY}	e_U	r
12	2	3	$(\sqrt{2})/2$
12	10	3	$(\sqrt{2})/2$
12	1	3	$(-1+\sqrt{3})/2$
12	1	9	$(-1+\sqrt{3})/2$
12	5	3	$(1+\sqrt{3})/2$
12	7	3	$(1+\sqrt{3})/2$
12	4	3	$(\sqrt{6})/2$
12	4	9	$(\sqrt{6})/2$

N	e_{XY}	e_U	r
12	4	1	$(3+\sqrt{33})/2$
12	4	5	$(-3+\sqrt{33})/2$
12	8	1	$(-3+\sqrt{33})/2$
12	8	5	$(3+\sqrt{33})/2$
48	16	21	$\sqrt{3}$
48	16	9	$\sqrt{3}$
48	32	3	$\sqrt{3}$
48	32	15	$\sqrt{3}$

The proof is completely parallel to Theorem 4.2.

Lemma 4.6. *If a solution to the Equation* (11) *has a sub-K-relation of length two, then the relation consists of terms* (12)*,* (56) *or* (78)*. Moreover, in this case all the solutions of Equation* (11) *are of the form* $(\zeta_{XY}, \zeta_U) = (\zeta_N^{e_{XY}}, \zeta_N^{e_U})$ *for some* $N \in \mathcal{N}(5)$*.*

Proof. As in Lemma 4.4 we use that

$$a_1\zeta_{XY}^{\alpha_1}\zeta_U^{\beta_1} + a_2\zeta_{XY}^{\alpha_2}\zeta_U^{\beta_2} = 0, \quad \text{implies} \quad -a_1/a_2 = \zeta_{XY}^{\alpha_2-\alpha_1}\zeta_U^{\beta_2-\beta_1} \in \{\pm 1\} \qquad (13)$$

Only the polynomials $\{-2, r(r-1), r(r+1), -2(r^2-1)\}$ appear as coefficients of the equation (11). If $a_1 \neq a_2$ then solving (13) for r gives a solution in $\mathbb{Q}$, except when $a_1 = -2(r^2-1)$ and $a_2 = -2$ (or vice versa). In this case $-a_1/a_2 = -(r^2-1) = -1$ and $r = \sqrt{2}$. From terms (37) and (48) we deduce the equations $\zeta_{XY}\zeta_U^2 + \zeta_{XY}^3\zeta_U^2 = 0$ and $\zeta_{XY}^2\zeta_U^2 + \zeta_U^2 = 0$ respectively. In these two cases $\zeta_{XY}^2 = -1$ and the equation reduces to $3\zeta_U^8 + 2\zeta_U^4 + 3 = 0$, which is possible only if $\frac{-1\pm 2\imath\sqrt{2}}{3}$ were a root of unity. A check over the finitely many roots of unity that are quadratic shows that this is impossible. The remaining combinations (38) and (47) imply $\zeta_{XY} = -1$ and then also $\zeta_U = \pm i$, contradicting the hypothesis of Theorem 4.5.

Hence it remains to treat the case where $a_1 = a_2$. The partition (34) implies that $\zeta_{XY} = -1$, contradiction. We discuss the remaining cases, the partitions (12), (56) and (78). Each of the remaining terms has a factor $(\zeta_U^2+1)(\zeta_U^2-1)$. If $\zeta_U^2 = -1$ then Equation (11) reduces to

$$\zeta_{XY}^3 + (2r^2-1)\zeta_{XY}^2 + (2r^2-1)\zeta_{XY} + 1 \; = \; 0.$$

If $r = \sqrt{2}/2$ then $\zeta_{XY}^3 = -1$, a solution that appears for $N = 6$, see the table in Theorem 4.5. Otherwise, none of the coefficients of this relation of length three is zero. Hence the relation is irreducible, any solution appears for $N \in \mathcal{N}(3)$. We analyze the remaining cases separately. They are

(1) $\zeta_{XY} + \zeta_{XY}^2\zeta_U^4 = 0$,
(2) $\zeta_{XY}^2 + \zeta_{XY}\zeta_U^4 = 0$,
(3) $\zeta_{XY}^3\zeta_U^2 + \zeta_U^2 = 0$.

In case (1) Equation (11) becomes, after taking out the factors we discussed,

$$2\zeta_U^8 + r(r+1)\zeta_U^6 - 2(r^2-2)\zeta_U^4 + r(r+1)\zeta_U^2 + 2 \; = \; 0.$$

In case (2) Equation (11) is the preceding equation, with r replaced by $-r$. Finally in case (3), Equation (11) becomes, after taking out the factors we discussed,

$$r^2(3r^2+1)\zeta_U^8 - 12r^2(r^2-1)\zeta_U^6 + 2r^2(9r^2-13)\zeta_U^4 - 12r^2(r^2-1)\zeta_U^2 + r^2(3r^2+1) \;=\; 0$$

If none of the coefficients is zero in any of the two preceding displayed equations, these are either irreducible (hence with a solution for ζ_U in for $N \in \mathcal{N}(5)$) or reducible with a partition in two plus three terms, hence with a solution for ζ_U in for $N \in \mathcal{N}(3)$. Since ζ_{XY} is determined, being a power of ζ_U, the claim follows in this case.

The only cases of a real quadratic number annihilating one of the coefficients are $r = \frac{\sqrt{2}}{2}$, $r = \frac{\sqrt{3}}{3}$ and $r = \frac{\sqrt{13}}{3}$. In each case some of the solutions lie on the unit circle, but a loop over the finitely many roots of unity of degree 16 over $\mathbb{Q}$ shows that these solutions are not roots of unity. □

Proof of Theorem 4.5, finiteness statement. The only triples (and four-tuples) that are not good are (labeling of the terms as they appear in (11)) $(1, 3, 6)$, $(2, 4, 5)$ and all the triples contained in $3, 4, 7, 8$ as well as the four-tuple $(3, 4, 7, 8)$. By the preceding lemma we only need to deal with atom-free partitions of (8) without a part of length two since the number of possibilities for $N \in \mathcal{N}(5)$ is finite. The only possibilities are $(5, 3)$, $(4, 4)$ and (8). The relation of length 8 is obviously good and by the preceding statement, if one of the relations of length 3 resp. 4 is bad, the complementary relation of length 5 or 3 is good. □

4.4. **Implementation for the case** $\mathrm{Prym}_3(2, 2)$**.** The basic algorithm of the preceding case applies here as well. We first test all the possible choices for (ζ_{XY}, ζ_U) corresponding to an irreducible relation. Equation (11) can also be written as a quadratic polynomial in r, the coefficients being polynomial in ζ_{XY} and ζ_U. Consequently, all the preceding remarks on whose to efficiently test the cases apply here as well.

The reduction step to the irreducible case works less well here, compared to the case $\mathrm{Prym}_3(2, 1, 1)$. In fact, there are 56 partitions into a triple and a relation of length 5 where the greatest common divisor of the two d associated by Lemma 4.3 with the triple and the 5-tuple is equal to two. Moreover, there are 64 partitions into two four-tuples such that the corresponding greatest common divisor is two and, finally, there are 6 such partitions where the greatest common divisor is equal four.

Not all of them can be ruled out as we did in Case (4,4) for $\mathrm{Prym}_3(2, 1, 1)$, besides the fact that this is not feasible, since the solutions with $N = 48$ stem from such cases. Instead, we loop over all $N \in \mathcal{N}(8) \cup 2 \cdot \mathcal{N}(4) \cup 4 \cdot \mathcal{N}(4)$, as stated in the theorem to cover all the cases.

5. Finiteness of the input data for the Thurston-Veech construction

In this section we complete the proof of the finiteness statements in Theorem 1.1 and Theorem 1.2. First, we determine the heights of the cylinders of any suitable direction and justify the norm condition we restricted ourselves to when tabulating the solutions to the relations in roots of unity. The second statement is a finiteness result for relative periods. The third step is to argue via the Thurston-Veech construction that finiteness holds.

5.1. **Heights.** The heights of the cylinders in a suitable direction are determined in the loci $\mathrm{Prym}_3(2,1,1)$ and $\mathrm{Prym}_3(2,2)$ by the widths, real multiplication and the Prym involution.

Lemma 5.1. *Normalizing $r_1 = 1$ and writing $r_2 = a + b\sqrt{D}$, the tuple heights of the cylinders is proportional to $(h_1 = 1, h_2, h_3 = h_1)$, where*

$$h_2 \;=\; \frac{2a + 2b\sqrt{D}}{Db^2 - a^2}.$$

Proof. We use the condition on the complex flux implied by complete periodicity, which states ([McM03, Theorem 4.5], see also [CS08]) that $\sum_{i=1}^{k} r_i h_i^\sigma = 0$ for any direction that completely decomposes into k cylinders. In our setting, this amounts to $2r_1 h_1^\sigma + r_2 h_2^\sigma = 0$ and this gives the above value. □

Corollary 5.2. *For any tuple $(r_1 = 1, r_2)$ of widths of cylinders in a suitable direction on a Veech surface in* $\mathrm{Prym}_3(2,1,1)$ *or* $\mathrm{Prym}_3(2,2)$ *we have $N_{\mathbb{Q}}^K(r_2) < 0$.*

Proof. Since $h_1/h_2 = -r_2^\sigma/2 > 0$ and since $r_2 > 0$ we deduce $N_{\mathbb{Q}}^K(r_2) < 0$. □

5.2. **Relative periods.** A saddle connection γ is said to *represent a relative period*, if it joins a simple zero to the double zero in the locus $\mathrm{Prym}_3(2,1,1)$ or if it joins the two double zeros in the locus $\mathrm{Prym}_3(2,2)$. Note that this is an abuse of terminology, since a saddle connection joining the two simple zeros does not qualify.

Proposition 5.3. *There exists a finite set $\mathcal{R} \subset \mathbb{P}(\mathbb{R}^3)$ such that for any Veech surface $(X,\omega) \in$* $\mathrm{Prym}_3(2,1,1)$ *or in* $\mathrm{Prym}_3(2,2)$*, any suitable direction Θ on X and any saddle connection γ representing a relative period the tuple $(r_1 : r_2 : |\gamma|)$ belongs to $\mathcal{R}$.*

Proof. We start with the case $\mathrm{Prym}_3(2,1,1)$. First, the tuple $(r_1 = 1, r_2, \zeta_{XY}, \zeta_U)$ is constrained in any suitable direction by Equation (8), to which there are only finitely many solutions by Theorem 4.2. We may assume that the torsion order N and one of the finitely many choices for $\zeta_{XY} = \zeta_N^{e_X}$ and $\zeta_U = \zeta_N^{e_U}$ as in the table in Theorem 4.2 are fixed. Using Equation (7), we see that this tuple also determines X, Y, U and hence ω_Θ up to finite ambiguity.

Let δ be the path along the real axis from $z = 0$ to $z = 1$. Direct computation gives:

$$\begin{aligned}\int_\delta \omega_\Theta \;&=\; r_1 \left[\log\frac{z-x}{z+x} - \log\frac{z-y}{z+y}\right]_0^1 + r_2\left[\frac{z-u}{z+u}\right]_0^1 \\ &=\; r_1 \log(\zeta_{XY}^{-1}) + r_2\log(\zeta_U) = -r_1\frac{e_X}{N} + r_2\frac{e_U}{N}.\end{aligned} \tag{14}$$

The saddle connection γ is a simple curve homotopic to δ up to a union of loops once around a subset of the points $\{x, y, -x, -y, u, -u\}$. The residue of ω_Θ around these points is $\pm r_1$ and $\pm r_2$ respectively. Since γ is disjoint from $\rho(\gamma)$ up to the endpoint, we do not need a loop both around x and $-x$ etc. We obtain

$$\int_\gamma \omega_\Theta \;=\; \int_\delta \omega_\Theta + kr_1 + \ell r_2, \quad k \in \{-2,-1,0,1,2\}, \quad \ell \in \{-1,0,1\}. \tag{15}$$

This is a finite list of possibilities.

In the case $\mathrm{Prym}_3(2,2)$ similarly the knowledge of the roots of unity given as the solutions of the equation in Theorem 4.5 determines the stable form by the calculation that lead to Equation (11).

The computation of the relative period (as already given in (10)) yields

$$\begin{aligned}\int_{-1}^{1} \omega_\Theta &= 2r_1 \log\left(\frac{(1-x)(1+y)}{(1+x)(1-y)}\right) + 2r_2 \log\left(\frac{1-u}{1+u}\right) \\ &= 2r_1 \log(\zeta_{XY}^{-1}) + 2r_2 \log(\zeta_U) = -2r_1 \frac{e_X}{N} + 2r_2 \frac{e_U}{N}\,.\end{aligned} \tag{16}$$

The same argument as above, based on the observation that δ is a simple curve gives here again that Equation (15) holds. □

5.3. **Finiteness of the possibilities for the intersection matrix.** We call a pair (Θ_1, Θ_2) of suitable directions on a Veech surface $(X, \omega) \in \mathrm{Prym}_3(2,1,1)$ *admissible*, if there exists a saddle connection β in the direction Θ_2 that represents a relative period and that crosses one of the cylinders of the direction Θ_1 just once and intersects no other cylinder of the direction Θ_1. Similarly, we call a Veech surface (X, ω) in $\mathrm{Prym}_3(2,2)$ *admissible*, if (Θ_1, Θ_2) are as above but now we ask that β crosses the fixed cylinder (called C_2) once and crosses no other cylinder *or* that β crosses both exchanged cylinders (called C_1 and C_3) once and crosses no other cylinder.

We call a Veech surface $(X, \omega) \in \mathrm{Prym}_3(2,1,1)$ or in $\mathrm{Prym}_3(2,2)$ *normalized*, if the horizontal and vertical direction are an admissible pair of suitable directions and if the cylinder C_1 (according to Convention 1) has width $r_1 = 1$ and $h_1 = 1$.

Lemma 5.4. *Any Veech surface* $(X, \omega) \in \mathrm{Prym}_3(2,1,1)$ *or in* $\mathrm{Prym}_3(2,2)$ *can be normalized.*

Proof. A suitable direction Θ_1 exists by Lemma 3.2 resp. Lemma 3.3. Now it suffices to take a cylinder in the direction Θ_1 with zero of different orders (resp. with two different double zeros) on its boundaries. Take β to be the saddle connection joining these zeros and Θ_2 to be the direction of β. Obviously the pair (Θ_1, Θ_2) is admissible. The normalization is an obvious consequence of the transitivity of the action of $\mathrm{GL}_2(\mathbb{R})$. □

We denote by Z_j, $j = 1, 2, 3$ the cylinders of the vertical direction, also labeled according to Convention 1. The goal of this section is the following finiteness statement for the intersection matrix.

Proposition 5.5. *Suppose that* (X, ω) *is normalized. There is only a finite number of possibilities for the heights and widths* $w(C_i), h(C_i), w(Z_i), h(Z_i)$ *of the cylinders in the horizontal and vertical direction as well as for the intersection matrix* $M = (M_{ij})$ *where* $M_{ij} = C_i \cdot Z_j$ *is the geometric intersection number of horizontal and vertical cylinders.*

As preparation, recall that the obvious properties

$$w(Z_j) = \sum_{i=1}^{3} M_{ij} h(C_i) \qquad \text{and} \qquad w(C_i) = \sum_{j=1}^{3} M_{ij} h(Z_j)$$

are expressed in terms of the matrix $M = (M_{ij})$ as

$$w(Z) \;=\; M^T \cdot h(C) \qquad \text{and} \qquad w(C) \;=\; M \cdot h(Z).$$

The point here is that M is never invertible. But we can defined a reduced intersection matrix, that turns out to be invertible. We define the reduced intersection matrix to be $M^{\rm red} = \left(\begin{smallmatrix} M_{11}+M_{13} & 2M_{12} \\ M_{21} & M_{22} \end{smallmatrix}\right)$, and since $M_{11} = M_{33}$, $M_{13} = M_{31}$, $M_{21} = M_{23}$ and $M_{12} = M_{32}$, the preceding equation reads

$$w(Z)^{\rm red} \;=\; (M^{\rm red})^T \cdot h(C)^{\rm red}, \qquad \text{where } \begin{pmatrix} a \\ b \\ a \end{pmatrix}^{\rm red} \;=\; \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right). \tag{17}$$

Proof of Proposition 5.5. Given a horizontal and a vertical irreducible direction, we may fix one of the finitely many choices for r_2 given by Proposition 5.3 and the heights are given by Lemma 5.1. The projective tuple $(w(Z_1) : w(Z_2) : |\beta|)$ is one of the finitely many tuples appearing according to Proposition 5.3. Since $|\beta|= 1$ or $|\beta|= h_2$ according to the definition of an admissible direction, it suffices to prove finiteness for a fixed tuple $(w(Z_1), w(Z_2), |\beta|)$. By Lemma 5.1 again this determines also the $h(Z_i)$.

We now use that the intersection numbers are integral, in particular invariant under Galois conjugation. The pairs $(w(C_1), w(C_2)$ and hence also $(w(Z_1), w(Z_2)$ are a $\mathbb{Q}$-basis of K. The same applies to the vertical direction. Consequently, the system

$$\begin{pmatrix} w(Z_1) & w(Z_1)^\sigma \\ w(Z_2) & w(Z_2)^\sigma \end{pmatrix} \;=\; (M^{\rm red})^T \cdot \begin{pmatrix} h(C_1) & h(C_1)^\sigma \\ h(C_2) & h(C_2)^\sigma \end{pmatrix}. \tag{18}$$

can be solved uniquely for the matrix $M^{\rm red}$, for any of the fixed possibilities for $(h_1, h_2, w(Z_1), w(Z_2))$. By integrality and positivity, there is only a finite number of possibilities for M_{11} and M_{13} given $M_{11}+M_{13}$. Together with the conditions stated at the beginning of the proof, this determines the intersection matrix completely. □

Proof of Theorem 1.1 and Theorem 1.2, finiteness statement. We recall that by the Thurston-Veech construction (see e.g. [McM06a]) a Veech surface with periodic horizontal and vertical directions of given heights $h(C_i)$ and $h(Z_j)$ is composed of rectangles of size $h(C_i) \times h(Z_j)$. The number of such rectangles is bounded by the total sum of the entries of the intersection matrix. All these quantities are finite by Proposition 5.5. □

5.4. **Implementation.** The finiteness proof above is constructive and can be implemented by performing in the case $\text{Prym}_3(2,1,1)$ for each of the fields $\mathbb{Q}(\sqrt{D_0})$ where[1] $D_0 \in \{2,3,6,33\}$ as appearing in table in Theorem 4.2 resp. in the case $\text{Prym}_3(2,2)$ for the $D_0 \in \{2,3,6,33\}$ as they appear in table in Theorem 4.5 the following steps.

(1) Compute the complete geometry in each suitable direction, i.e. for every solution according to the table with the given D_0 store the possible triples $(r_1 = 1, r_2, \gamma)$ in a list $\mathcal{R}$, where

$$|\gamma|= -\frac{e_X}{N} + r_2\frac{e_U}{N} + k + \ell r_2, \quad \text{for} \quad k \in \{-2,-1,0,1,2\}, \quad \ell \in \{-1,0,1\}$$

[1]Note that so far we have not been specifying the order of real multiplication, just the field and we specify it by a square-free integer D_0.

for $\mathrm{Prym}_3(2,1,1)$ and

$$|\gamma| = -2\frac{e_X}{N} + 2r_2\frac{e_U}{N} + k + \ell r_2, \quad \text{for} \quad k \in \{-2,-1,0,1,2\}, \quad \ell \in \{-1,0,1\}$$

for $\mathrm{Prym}_3(2,2)$ respectively. Moreover, γ satisfies $0 < |\gamma| < \max\{1, r_2\}$, since we may assume that γ is a saddle connection on the boundary of one of the cylinders. Store also (h_1, h_2) computed according to Lemma 5.1.

(2) For any pair of tuples in $\mathcal{R}$ normalize the first element of the pair to be equal to $(r_1 = 1, r_2, h_1 = 1, h_2)$ (forgetting about the saddle connection) and normalize the second tuple to be $(w(Z_1), w(Z_2), |\beta|, h(Z_1), h(Z_2))$ with either

$$|\beta| = h_1 \qquad \text{or} \qquad |\beta| = h_2$$

corresponding to the cases if $|\beta|$ crosses C_1 or C_2 for $\mathrm{Prym}_3(2,1,1)$ and

$$|\beta| = 2h_1 \qquad \text{or} \qquad |\beta| = h_2$$

corresponding to the cases if $|\beta|$ crosses C_1 or C_2 for $\mathrm{Prym}_3(2,2)$ respectively. Compute M^{red} according to (18) and store the matrix if it is non-negative integral and $(M^{\mathrm{red}})_{12} \equiv 0 \mod 2$.

The list of cases for $\mathrm{Prym}_3(2,1,1)$ is reduced even more by the following constraint, that is easily checked using the complete list of configurations given in Figure 1.

Lemma 5.6. *Suppose that $(X,\omega) \in \mathrm{Prym}_3(2,1,1)$ has a horizontal direction, normalized according to Convention 1. Then there is a suitable direction with a saddle connection β contained in C_1 crossing this cylinder once.*

The corresponding statement for $\mathrm{Prym}_3(2,2)$ is even easier to obtain. Note that C_2, being fixed by the Prym involution, contains a fixed point of the Prym involution in its interior. This proves the following lemma.

Lemma 5.7. *Suppose that $(X,\omega) \in \mathrm{Prym}_3(2,2)$ has a horizontal direction, normalized according to Convention 1. Then there is a suitable direction with a saddle connection β contained in C_2 crossing this cylinder once.*

6. Implementing the algorithm

6.1. **Results in the case** $\mathrm{Prym}_3(2,1,1)$**.** Implementing the algorithm of Section 5.4 shows that there is no solutions for $D_0 \in \{2,3,6\}$. Moreover, for $D_0 = 33$ there are 3 possible matrices for M^{red}. Two matrices correspond to the case where β crosses the cylinder C_2. These can be excluded by Lemma 5.6. The remaining case is given by

$$w(C_1) = 1, \qquad h(C_1) = 1, \qquad w(Z_1) = \tfrac{9+\sqrt{33}}{2}, \qquad h(Z_1) = \tfrac{-3+\sqrt{33}}{36}$$

$$w(C_2) = \tfrac{3+\sqrt{33}}{6}, \qquad h(C_2) = \tfrac{3+\sqrt{33}}{2}, \qquad w(Z_2) = \tfrac{21+3\sqrt{33}}{2} \qquad h(Z_2) = \tfrac{1}{3}$$

and the reduced intersection matrix

$$M^{\mathrm{red}} = \begin{pmatrix} 0 & 6 \\ 3 & 3 \end{pmatrix}. \quad \text{Hence} \quad M = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 3 & 3 \\ 0 & 3 & 0 \end{pmatrix}.$$

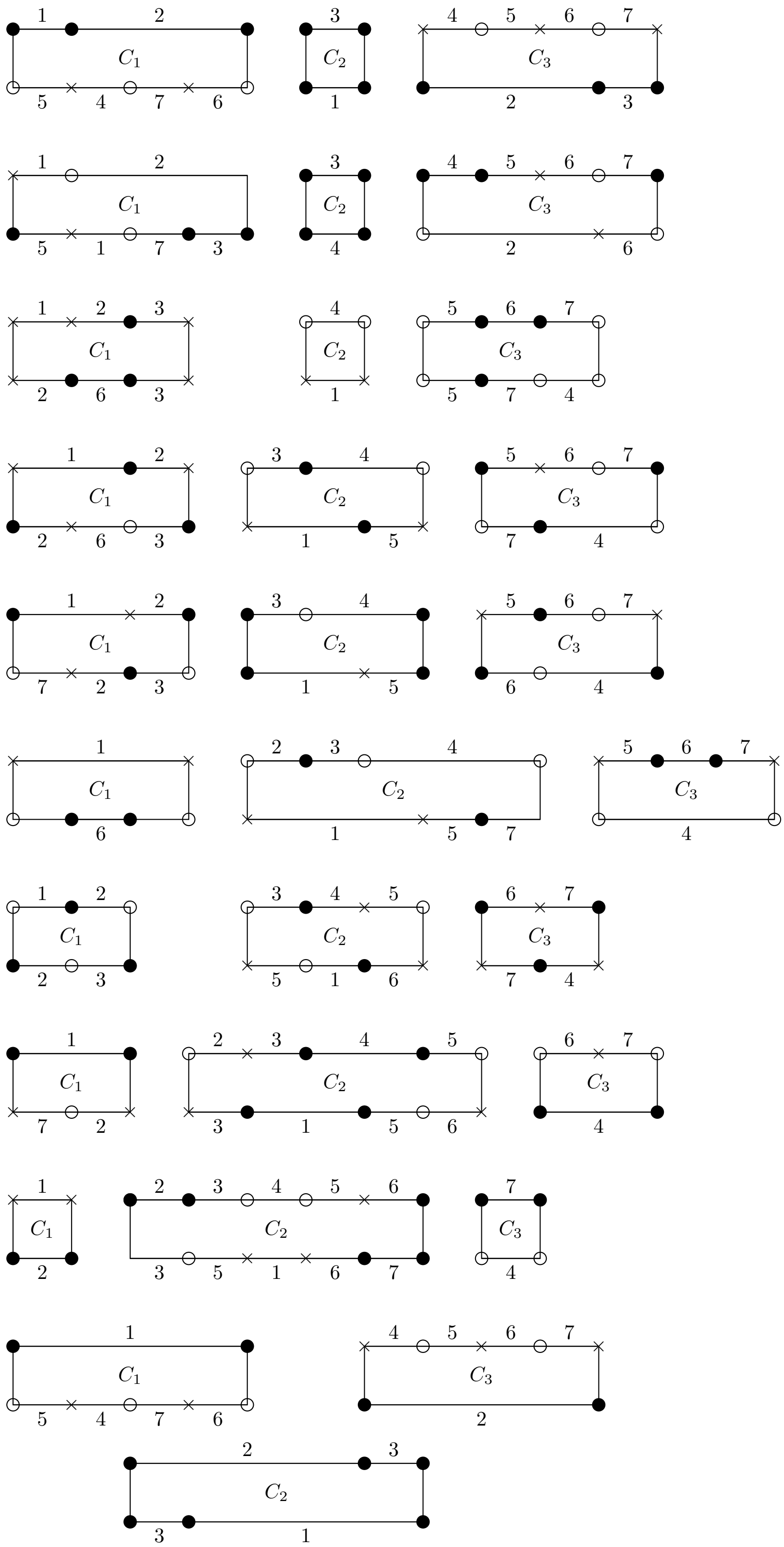

Figure 1. List of possible separatrix diagrams in a suitable direction, case $\mathrm{Prym}_3(2, 1, 1)$.

6.2. **Results in the case** $\mathrm{Prym}_3(2,2)$**.** The preceding algorithm shows that there is no solutions for $D_0 = 6$. Moreover, for $D_0 \in \{2,3,33\}$ there are 7 possible matrices for M^{red}. More precisely the following proposition holds:

Proposition 6.1. *If* (Θ_1,Θ_2) *is a pair of directions on a primitive Veech surface in* $\mathrm{Prym}_3(2,2)$ *satisfying above convention then the possible reduced intersection matrices are:*

K	M^{red}	r_2^{hor}	K	M^{red}	r_2^{hor}
$\mathbb{Q}[\sqrt{2}]$	$\begin{pmatrix} 72 & 48 \\ 24 & 18 \end{pmatrix}$	$(\sqrt{2})/2$	$\mathbb{Q}[\sqrt{33}]$	$\begin{pmatrix} 6 & 24 \\ 12 & 54 \end{pmatrix}$	$(3+\sqrt{33})/2$
$\mathbb{Q}[\sqrt{3}]$	$\begin{pmatrix} 72 & 24 \\ 12 & 6 \end{pmatrix}$	$(-1+\sqrt{3})/2$	$\mathbb{Q}[\sqrt{33}]$	$\begin{pmatrix} 6 & 24 \\ 3 & 18 \end{pmatrix}$	$(-3+\sqrt{33})/2$
$\mathbb{Q}[\sqrt{3}]$	$\begin{pmatrix} 72 & 24 \\ 48 & 18 \end{pmatrix}$	$(1+\sqrt{3})/2$	$\mathbb{Q}[\sqrt{33}]$	$\begin{pmatrix} 3 & 6 \\ 3 & 0 \end{pmatrix}$	$(-3+\sqrt{33})/2$
$\mathbb{Q}[\sqrt{3}]$	$\begin{pmatrix} 36 & 12 \\ 30 & 12 \end{pmatrix}$	$\sqrt{3}$			

We now discuss case by case each possible intersection matrix.

6.3. **Solutions for the intersection matrices when** $D_0 = 2$**.** We consider the reduced intersection matrix $M_1^{\mathrm{red}} = \left(\begin{smallmatrix} 72 & 48 \\ 24 & 18 \end{smallmatrix}\right)$.

$$\begin{array}{llll} w(C_1) = 1, & h(C_1) = 1, & w(Z_1) = 72+48\sqrt{2}, & h(Z_1) = \frac{3-2\sqrt{2}}{24} \\ w(C_2) = \frac{\sqrt{2}}{2}, & h(C_2) = 2\sqrt{2}, & w(Z_2) = 48+36\sqrt{2} & h(Z_2) = \frac{-4+3\sqrt{2}}{12} \end{array}$$

6.4. **Solutions for the intersection matrices when** $D_0 = 3$**.** For the reduced intersection matrix $M_2^{\mathrm{red}} = \left(\begin{smallmatrix} 72 & 24 \\ 12 & 6 \end{smallmatrix}\right)$ one has

$$\begin{array}{llll} w(C_1) = 1, & h(C_1) = 1, & w(Z_1) = 48+24\sqrt{3}, & h(Z_1) = \frac{2-\sqrt{3}}{24} \\ w(C_2) = \frac{-1+\sqrt{3}}{2}, & h(C_2) = -2+2\sqrt{3}, & w(Z_2) = 12+12\sqrt{3} & h(Z_2) = \frac{-5+3\sqrt{3}}{12}. \end{array}$$

For the reduced intersection matrix $M_3^{\mathrm{red}} = \left(\begin{smallmatrix} 72 & 24 \\ 48 & 18 \end{smallmatrix}\right)$ one has

$$\begin{array}{llll} w(C_1) = 1, & h(C_1) = 1, & w(Z_1) = 168+96\sqrt{3}, & h(Z_1) = \frac{2-\sqrt{3}}{24} \\ w(C_2) = \frac{1+\sqrt{3}}{2}, & h(C_2) = 2+2\sqrt{3}, & w(Z_2) = 60+36\sqrt{3} & h(Z_2) = \frac{-5+3\sqrt{3}}{12}. \end{array}$$

For the reduced intersection matrix $M_4^{\mathrm{red}} = \left(\begin{smallmatrix} 36 & 12 \\ 30 & 12 \end{smallmatrix}\right)$ one has

$$\begin{array}{llll} w(C_1) = 1, & h(C_1) = 1, & w(Z_1) = 36+20\sqrt{3}, & h(Z_1) = \frac{2-\sqrt{3}}{12} \\ w(C_2) = \sqrt{3}, & h(C_2) = 2(\sqrt{3})/3, & w(Z_2) = 12+8\sqrt{3}, & h(Z_2) = \frac{-5+3\sqrt{3}}{6}. \end{array}$$

6.5. **Solutions for the intersection matrices when** $D_0 = 33$**.** For the reduced intersection matrix $M_5^{\mathrm{red}} = \left(\begin{smallmatrix} 6 & 24 \\ 12 & 54 \end{smallmatrix}\right)$ one has

$$\begin{array}{llll} w(C_1) = 1, & h(C_1) = 1, & w(Z_1) = 12+2\sqrt{33}, & h(Z_1) = \frac{6-\sqrt{33}}{6}, \\ w(C_2) = \frac{3+\sqrt{33}}{2}, & h(C_2) = \frac{3+\sqrt{33}}{6} & w(Z_2) = 51+9\sqrt{33}, & h(Z_2) = \frac{-5+\sqrt{33}}{12}. \end{array}$$

For the reduced intersection matrix $M_6^{\rm red} = \left(\begin{smallmatrix} 6 & 24 \\ 3 & 18 \end{smallmatrix}\right)$ one has

$$\begin{array}{llll} w(C_1) = 1, & h(C_1) = 1, & w(Z_1) = \frac{9+\sqrt{33}}{2} & h(Z_1) = \frac{6-\sqrt{33}}{6} \\ w(C_2) = \frac{-3+\sqrt{33}}{2}, & h(C_2) = \frac{-3+\sqrt{33}}{6}, & w(Z_2) = 15+3\sqrt{33}, & h(Z_2) = \frac{-5+\sqrt{33}}{12}. \end{array}$$

For the reduced intersection matrix $M_7^{\rm red} = \left(\begin{smallmatrix} 3 & 6 \\ 3 & 0 \end{smallmatrix}\right)$ one has

$$\begin{array}{llll} w(C_1) = 1, & h(C_1) = 1, & w(Z_1) = \frac{3+\sqrt{33}}{2}, & h(Z_1) = \frac{-3+\sqrt{33}}{12} \\ w(C_2) = \frac{-3+\sqrt{33}}{2}, & h(C_2) = \frac{-3+\sqrt{33}}{6} & w(Z_2) = 6, & h(Z_2) = \frac{7-\sqrt{33}}{12}. \end{array}$$

7. Non-existence in $\mathrm{Prym}_3(2,1,1)$

We can now complete the proof of non-existence, using again the list of configurations in this stratum.

Proof of Theorem 1.2. From the intersection matrix we deduce that C_1 and C_3 consists of intersection points with Z_2 only. Since 9 is odd, the cylinder C_2 has a fixed point of ρ in the center of one of the rectangles, necessarily an intersection with Z_2. In Figure 2 this is the leftmost rectangle of the middle strip. In C_2 the other two intersection rectangles with Z_2 are symmetric with respect to this rectangle. Suppose with loss of generality that there is a double zero on the bottom C_1.

We now use that no possibility for the relative period between a simple zero and a double zero (compare the list in the previous section) is rational. Consequently, the lower boundary of C_1 does not contain a simple zero. By inspection of Figure 1 we conclude that the lower boundary of C_1 has a single saddle connection. This implies that the three occurrences of Z_2-rectangles in C_2 are adjacent, as drawn in Figure 2. To ensure that the total angle at the double zero (indicated by a black

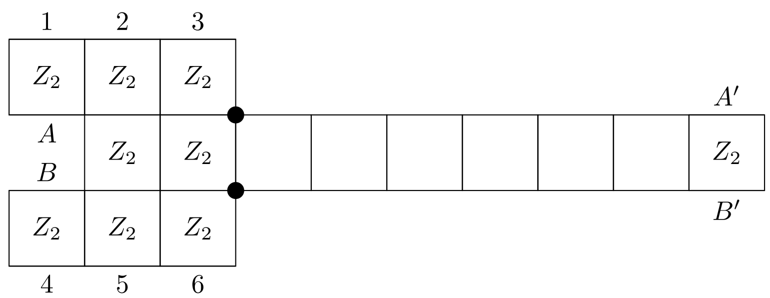

Figure 2. Ruling out the remaining case in $\mathrm{Prym}_3(2,1,1)$.

circle) does not exceed 6π the two rectangles with the symbol 1 resp. 2 have to be glued. This contradicts that the unlabeled rest of C_2 consists of just two vertical cylinders Z_1 and Z_3. □

8. Effective finiteness in the locus $\mathrm{Prym}_3(2,2)$

In this section we complete the proof of Theorem 1.1 by giving a practically feasible algorithm to compile a short finite list of remaining candidate surfaces. The algorithm proceeds case by case according to the list of possible separatrix diagrams in Figure 4. We give all the details for a specific case, the fourth diagram 'SD4' in this figure, see also the horizontal direction of the surface in Figure 3 below. We summarize the output of the algorithm in the remaining cases.

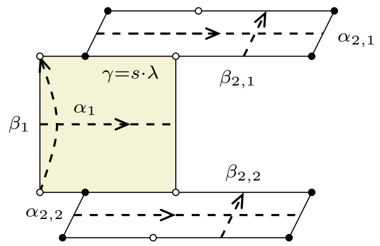

Figure 3. If $\alpha_2 := \alpha_{2,1} + \alpha_{2,2}$ and $\beta_2 := \beta_{2,1} + \beta_{2,2}$ then $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ is a symplectic basis of $H_1(X, \mathbb{Z})^-$.

8.1. Finding arithmetic surfaces with a given reduced intersection matrix.

We have already seen that the finiteness of the possibilities for the parameters $w(C_i), h(C_i), w(Z_i), h(Z_i)$ as well as the reduced intersection matrix gives at most a finite number of possibilities for the flat surface (X, ω). We also know that (X, ω) is tiled by rectangles $R_{i,j}$ consisting of t he intersection of the cylinders C_i and Z_j for $i, j \in \{1, 2\}$.

To actually give a good practical upper bound for the number of possibilities we proceed as follows. First, we use a prototype to present the surface with the given separatrix diagrams in the horizontal direction in a standard form. Second, we loop over possible 'arithmetic surfaces' that encode the adjacency of the rectangles $R_{i,j}$.

The first step is given by the following proposition, whose proof is completely parallel to that of [LN14, Proposition 4.2].

Proposition 8.1. *Let $(X, \omega) \in \Omega E_D(2,2)$ be a Prym eigenform with horizontal separatrix diagram as in 'SD4', equipped with loops $\alpha_{i,j}$ and $\beta_{i,j}$ as presented in Figure 3. Then after applying a suitable element in the upper triangular group there exist $(w, h, t, e) \in \mathbb{Z}^4$ and $s \in (0,1)$ such that*

- *the tuple (w, h, t, e) satisfies* $(\mathcal{P}_D) := \begin{cases} w > 0, h > 0,\ 0 \le t < \gcd(w,h), \\ \gcd(w,h,t,e) = 1, \\ D = e^2 + 8wh, \\ 0 < \lambda := \frac{e+\sqrt{D}}{2} < w \end{cases}$,
- *there exists a generator T of $\mathcal{O}_D$ with the property $T^*(\omega) = \lambda\omega$ that is represented in the basis $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ by* $\begin{pmatrix} e & 0 & 2w & 2t \\ 0 & e & 0 & 2h \\ h & -t & 0 & 0 \\ 0 & w & 0 & 0 \end{pmatrix}$.

- *and such that with this choice coordinates*

$$\begin{cases} \omega(\mathbb{Z}\alpha_1 + \mathbb{Z}\beta_1) = \lambda \cdot \mathbb{Z}^2, \\ \omega(\mathbb{Z}\alpha_{2,1} + \mathbb{Z}\beta_{2,2}) = \omega(\mathbb{Z}\alpha_{2,2} + \mathbb{Z}\beta_{2,2}) = \mathbb{Z}(w,0) + \mathbb{Z}(t,h), \\ \omega(\gamma) = (s\lambda, 0). \end{cases}$$

Conversely, let $(X,\omega) \in \mathrm{Prym}_3(2,2)$ *having the above decomposition such that there exists* $(w,h,t,e) \in \mathbb{Z}^4$ *verifying* $(\mathcal{P}_D)$ *such that, after normalizing by* $\mathrm{GL}^+(2,\mathbb{R})$*, the conditions are satisfied, then* $(X,\omega) \in \Omega E_D(2,2)$.

We refer to the parameters s and t as the *slit* and *twist* respectively.

For the second step we construct all square-tiled surfaces in $\mathrm{Prym}_3(2,2)$ with the horizontal separatrix diagram as in 'SD4' and such that the associated intersection matrix is a given matrix M^{red} as listed in Section 6.2. More concretely, let a_i be the number of squares of the horizontal cylinder C_i and b_j be the number of squares of the vertical cylinder Z_j. Then obviously

$$\begin{array}{ll} a_1 = M_{11}^{\mathrm{red}} + \frac{1}{2} M_{12}^{\mathrm{red}} & a_2 = 2 \cdot M_{21}^{\mathrm{red}} + M_{22}^{\mathrm{red}} \\ b_1 = M_{11}^{\mathrm{red}} + M_{21}^{\mathrm{red}} & b_2 = M_{12}^{\mathrm{red}} + M_{22}^{\mathrm{red}} \end{array}$$

Let $\ell_i \in \mathbb{N}$ for $i = 0, \ldots, 6$ be the length of the horizontal saddle connections γ_i with label i in Figure 4 (still diagram 'SD4') in the corresponding square-tiled surface. We determine all possible square-tiled surfaces by running a loop over all $\ell_3 \in [0..\min(a_1, a_2)]$, all twist parameters $T_1 \in [0..a_1]$, and all $T_2 \in [0..a_2]$ (in the cylinder C_i, with respect to the leftmost singularities in the figure) and then checking whether the vertical direction on the surface produces the given reduced intersection matrix. We refer to the square-tiled surfaces constructed in this way as the *arithmetic surfaces* underlying $(X,\omega) \in \Omega E_D(2,2)$ with suitable horizontal and vertical direction.

In order to convert such an arithmetic surface into a candidate for a Veech surface in $\Omega E_D(2,2)$ in the normalization given in Proposition 8.1, we replace each square by a rectangle $R_{i,j}$ according to the horizontal and vertical cylinder the square lies in. Next, we convert the twists T_i on the arithmetic surface into twists on the candidate for a Veech surface. In fact, if T_k shifts by $N_{i,j}^k$ squares that correspond to rectangles of type $R_{i,j}$, then obviously

$$t_k \;=\; h(Z_1) \cdot \sum_{j=1}^{3} (N_{j,1}^k + N_{j,3}^k) + h(Z_2) \cdot \sum_{j=1}^{3} N_{j,2}^k \,.$$

Finally, we scale by an upper triangular matrix so that the central horizontal cylinder becomes a square.

Once the surface is constructed, we check that the vertical direction is admissible *i.e.* there is an invariant saddle connection that represents a relative period and that is contained in C_2. This rules out in practice a large number of surfaces.

8.2. **Output of the algorithm.** For instance for the first intersection matrix M_1^{red} and the trace field $\mathbb{Q}[\sqrt{2}]$, we found 228 arithmetic surfaces, hence 228 candidates surfaces. Only 6 solutions have a vertical admissible direction. Some of them give the same prototype, in fact there are three different prototype, as listed in Table 1. The other intersection matrices are treated in the same way.

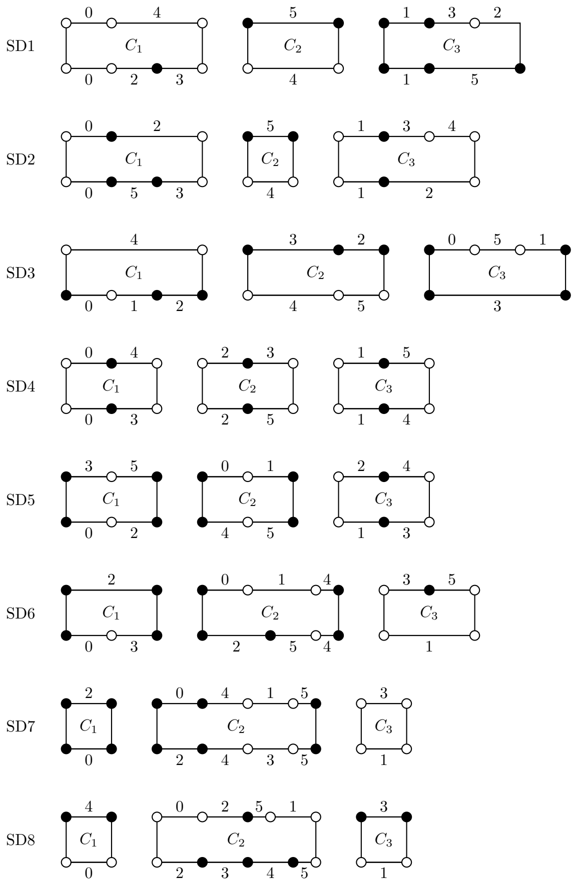

FIGURE 4. List of possible separatrix diagrams in a suitable direction, case $\mathrm{Prym}_3(2,2)$.

Remark 8.2. In the examples where the twist parameter is zero, one can additionally check the rationality constraint of ratios of moduli of vertical cylinders (see [McM06b, Theorem 6.3] for similar computations), since the cylinder decomposition in the vertical direction is easily computed. In the three examples in the preceding table with twist zero the moduli are indeed not commensurable, thus ruling out these 3 cases.

In view of the above remark and the results of the table, one concludes that there are at most 12 Teichmüller curves for which there is a translation surface having a cylinder decomposition with separatrix diagram SD4 of Figure 4.

For each of the 7 remaining possible cylinder decomposition in Figure 4 we apply the algorithm described above. The results are presented in slightly more condensed form in Table 2 below. A quick inspection of this table (combined with Remark 8.2) reveals the 92 remaining cases which concludes the proof of Theorem 1.1.

Reduced matrix	# Arithm. surf.	Prototypes (w,h,t,e)	slits	disc.
$\left(\begin{smallmatrix} 72 & 48 \\ 24 & 18 \end{smallmatrix}\right)$	228	$(4,1,0,0)$ $(12,3,1,0)$ $(12,3,2,0)$	$\frac{3+2\sqrt{2}}{6}$ $\frac{3+2\sqrt{2}}{6}$ $\frac{3+2\sqrt{2}}{6}$	32 288 288
$\left(\begin{smallmatrix} 72 & 24 \\ 12 & 6 \end{smallmatrix}\right)$	32	$(4,1,0,-4)$ $(12,3,1,-12)$ $(12,3,2,-12)$	$\frac{4+\sqrt{3}}{6}$ $\frac{4+\sqrt{3}}{6}$ $\frac{4+\sqrt{3}}{6}$	48 432 432
$\left(\begin{smallmatrix} 72 & 24 \\ 48 & 18 \end{smallmatrix}\right)$	336	$(4,1,0,4)$ $(12,3,1,12)$ $(12,3,2,12)$	$\frac{\sqrt{3}}{6}$ $\frac{\sqrt{3}}{6}$ $\frac{\sqrt{3}}{6}$	48 432 432
$\left(\begin{smallmatrix} 36 & 12 \\ 30 & 12 \end{smallmatrix}\right)$	180	$(6,9,1,0)$ $(6,9,2,0)$ $(6,9,1,0)$ $(6,9,2,0)$ $(12,18,1,0)$ $(12,18,5,0)$	$\frac{6-\sqrt{3}}{18}$ $\frac{6-\sqrt{3}}{18}$ $\frac{6+\sqrt{3}}{18}$ $\frac{6+\sqrt{3}}{18}$ $\frac{1+\sqrt{3}}{6}$ $\frac{1+\sqrt{3}}{6}$	432 432 432 432 1728 1728
$\left(\begin{smallmatrix} 6 & 24 \\ 12 & 54 \end{smallmatrix}\right)$	24	no solutions		
$\left(\begin{smallmatrix} 6 & 24 \\ 3 & 18 \end{smallmatrix}\right)$	0	no solutions		
$\left(\begin{smallmatrix} 3 & 6 \\ 3 & 0 \end{smallmatrix}\right)$	0	no solutions		

Table 1. Number of arithmetic candidate surfaces and prototypes with separatrix diagram SD4.

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Reduced matrix \ SD	1	2	3	4	5	6	7	8
$M_1^{\text{red}} = \left(\begin{smallmatrix} 72 & 48 \\ 24 & 18 \end{smallmatrix}\right)$	520 **12**	176 **1**	-	228 **3**	342 **4**	-	-	-
$M_2^{\text{red}} = \left(\begin{smallmatrix} 72 & 24 \\ 12 & 6 \end{smallmatrix}\right)$	108 **8**	63 **0**	-	32 **3**	88 **0**	-	-	-
$M_3^{\text{red}} = \left(\begin{smallmatrix} 72 & 24 \\ 48 & 18 \end{smallmatrix}\right)$	-	-	23 **0**	336 **3**	290 **8**	186 **3**	-	-
$M_4^{\text{red}} = \left(\begin{smallmatrix} 36 & 12 \\ 30 & 12 \end{smallmatrix}\right)$	-	-	0 **0**	180 **6**	48 **0**	214 **8**	-	-
$M_5^{\text{red}} = \left(\begin{smallmatrix} 6 & 24 \\ 12 & 54 \end{smallmatrix}\right)$	-	-	-	24 **0**	-	124 **7**	392 **18**	210 **20**
$M_6^{\text{red}} = \left(\begin{smallmatrix} 6 & 24 \\ 3 & 18 \end{smallmatrix}\right)$	-	-	0 **0**	0 **0**	0 **0**	0 **0**	-	-
$M_7^{\text{red}} = \left(\begin{smallmatrix} 3 & 6 \\ 3 & 0 \end{smallmatrix}\right)$	-	-	-	0 **0**	0 **0**	-	-	-
# Candidates	**20**	**1**	**0**	**15**	**12**	**18**	**18**	**20**

TABLE 2. Candidates for Teichmüller discs for each model. In bold we have indicated the number of candidate surfaces with an admissible vertical direction.

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