Integral Geometry of Tensor Valuations – recent results

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Abstract In this survey, we explain in an informal way recently introduced algebraic structures on the space of translation invariant, smooth tensor valuations, including convolution, product, Poincaré duality and Alesker-Fourier transform, and their relation to kinematic formulae for tensor valuations. We also discuss the connection to integral geometric formulae for area measures. Furthermore, we describe how the algebraic viewpoint leads to new intersectional kinematic formulae and substantially simplified Crofton formulae, for translation invariant tensor valuations.

1 Tensor Valuations

This chapter is based on the general introduction to valuations in Chap. 1, the description of tensor valuations in Chap. 2, and on the algebraic framework for scalar valuations provided in Chap. 4. There the relevant background information is provided, including references to previous work, motivation and hints to applications. The latter are also discussed in other parts of this volume (see especially Chaps. 10, 12 and 13).

Although we mainly consider translation invariant tensor valuations, we briefly recall the general definitions and relate them to the notation used in the translation invariant case and, in particular, in Chap. **??** which is devoted to Crofton formulae for tensor-valued curvature measures.

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1.1 Examples of Tensor Valuations

For $k \in \{0,\dots,n-1\}$ and $K \in \mathcal{K}^n$, let $\Lambda_0(K,\cdot),\dots,\Lambda_{n-1}(K,\cdot)$ denote the support measures associated with K. They are Borel measures on $\Sigma^n := \mathbb{R}^n \times \mathbb{S}^{n-1}$ which are concentrated on the normal bundle $\mathbf{nc}\, K$ of K. Let κ_n denote the volume of the unit ball and $\omega_n = n\kappa_n$ the volume of its boundary, the unit sphere. Using the support measures, we recall from Sect. 2.1 that the Minkowski tensors are defined by

$$\Phi_k^{r,s}(K) = \frac{1}{r!s!}\frac{\omega_{n-k}}{\omega_{n-k+s}}\int_{\Sigma^n} x^r u^s \,\Lambda_k(K, \mathrm{d}(x,u)),$$

for $k \in \{0,\dots,n-1\}$ and $r,s \in \mathbb{N}_0$, and

$$\Phi_n^{r,0}(K) := \frac{1}{r!}\int_K x^r \,\mathrm{d}x.$$

In addition, we define $\Phi_k^{r,s} := 0$ for all other choices of indices. Clearly, the tensor valuations $\Phi_k^{0,s}$ and $\Phi_n^{0,0}$, which are obtained by choosing $r = 0$, are translation invariant. However, these are not the only translation invariant examples, since e.g. $\Phi_{k-1}^{1,1}$, for $k \in \{1,\dots,n\}$, is also translation invariant.

Further examples of continuous, isometry covariant tensor valuations are obtained by multiplying the Minkowski tensors with powers of the metric tensor Q and by taking linear combinations. As shown by Alesker [1, 2], no other examples exist (see also Theorem 2.5). In the following, we write

$$\begin{aligned}\Phi_k^s(K) :&= \Phi_k^{0,s}(K) = \frac{1}{s!}\frac{\omega_{n-k}}{\omega_{n-k+s}}\int_{\Sigma^n} u^s \,\Lambda_k(K,\mathrm{d}(x,u))\\ &= \binom{n-1}{k}\frac{1}{\omega_{n-k+s}s!}\int_{\mathbb{S}^{n-1}} u^s \,S_k(K,\mathrm{d}u),\end{aligned}$$

for $k \in \{0,\dots,n-1\}$, where we used the kth area measure $S_k(K,\cdot)$ of K, a Borel measure on $\mathbb{S}^{n-1}$ defined by

$$S_k(K,\cdot) = \frac{n\kappa_{n-k}}{\binom{n}{k}}\Lambda_k(K,\mathbb{R}^n \times \cdot).$$

In addition, we define $\Phi_n^0 := V_n$ and $\Phi_n^s := 0$ for $s > 0$. The normalization is such that $\Phi_k^0 = V_k$, for $k \in \{0,\dots,n\}$, where V_k (also denoted by μ_k) is the kth intrinsic volume. Clearly, the tensor valuations $Q^i\Phi_k^s$, for $k \in \{0,\dots,n\}$ and $i,s \in \mathbb{N}_0$, are continuous, translation invariant, $\mathrm{O}(n)$-covariant, homogeneous of degree k and have tensor rank $2i+s$. If $k = n$, then necessarily $s = 0$, and if $k = 0$, then $\Phi_0^s(K)$ is independent of K. Hence, we usually exclude these trivial cases. Apart from these, Alesker showed that for each fixed $k \in \{1,\dots,n-1\}$ the valuations

$$Q^i\Phi_k^s, \quad i,s \in \mathbb{N}_0, 2i+s = p, s \neq 1,$$

form a basis of the vector space of all continuous, translation invariant, $\mathrm{O}(n)$-covariant tensor valuations of rank p which are homogenous of degree k. The fact that these valuations span the corresponding vector space is implied by [1, Prop. 4.9] (and [2]), the proof is based in particular on basic representation theory. A result of Weil [16, Thm. 3.5] states that differences of area measure of order k, for any fixed $k \in \{1, \dots, d-1\}$, are dense in the vector space of differences of finite, centered Borel measures on the unit sphere. From this the asserted linear independence of the tensor valuations can be inferred.

The situation for general tensor valuations (not necessarily translation invariant) is more complicated. As explained in Chap. 2, the valuations $Q^i \Phi_k^{r,s}$ span the corresponding vector space, but there exist linear dependences between these functionals. Although all linear relations are known and the dimension of the corresponding vector space (for fixed rank and degree of homogeneity) has been determined, the situation here is not perfectly understood.

In the following, it will often (but not always) be sufficient to neglect the metric tensor powers Q^i and just consider the tensor valuations Φ_k^s, since the metric tensor commutes nicely with the algebraic operations to be considered.

1.2 Integral Geometric Formulas

Let $A(n,k)$, $k \in \{0, \dots, n\}$, denote the affine Grassmannian of k-flats in $\mathbb{R}^n$, and let μ_k denote the motion invariant measure on $A(n,k)$ normalized as in [12, 13]. The Crofton formulas to be discussed below relate the integral mean

$$\int_{A(n,k)} \Phi_j^{r,s}(K \cap E)\, \mu_k(\mathrm{d}E)$$

of the tensor valuation $\Phi_j^{r,s}(K \cap E)$ of the intersection of K with flats $E \in A(n,k)$ to tensor valuations of K. Guessing from the scalar case, one would expect that only tensor valuations $\Phi_{n-k+j}^{r',s'}(K)$ are required. It turns out, however, that for general r the situation is more involved.

The following Crofton formulas for Minkowski tensors were established in [8]. Since $\Phi_j^{r,s}(K \cap E) = 0$ if $k < j$, we only have to consider the cases where $k \geq j$.

We start with the basic case $k = j$, which has a simple form.

Theorem 2.1. *For $K \in \mathcal{K}^n$, $r, s \in \mathbb{N}_0$ and $0 \leq k \leq n-1$,*

$$\int_{A(n,k)} \Phi_k^{r,s}(K \cap E)\, \mu_k(\mathrm{d}E) = \begin{cases} \tilde{\alpha}_{n,k,s}\, Q^{\frac{s}{2}}\, \Phi_n^{r,0}(K), & \textit{if } s \textit{ is even}, \\ 0, & \textit{if } s \textit{ is odd}, \end{cases}$$

where

$$\tilde{\alpha}_{n,k,s} = \frac{1}{(4\pi)^{\frac{s}{2}} \left(\frac{s}{2}\right)!} \frac{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n-k+s}{2}\right)}{\Gamma\left(\frac{n+s}{2}\right) \Gamma\left(\frac{n-k}{2}\right)}.$$

This result essentially follows from Fubini's theorem, combined with a relation due to McMullen, which connects the Minkowski tensors of $K \cap E$ and the Minkowski tensors of $K \cap E$ but with respect to the flat E as the ambient space (see (2) for a precise statement).

The main case $j < k$ is stated in the next theorem.

Theorem 2.2. *Let $K \in \mathscr{K}^n$ and $k, j, r, s \in \mathbb{N}_0$ with $0 \leq j < k \leq n-1$. Then*

$$\int_{A(n,k)} \Phi_j^{r,s}(K \cap E)\, \mu_k(\mathrm{d}E) = \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \chi^{(1)}_{n,j,k,s,z} Q^z \Phi^{r,s-2z}_{n+j-k}(K) + \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor -1} \chi^{(2)}_{n,j,k,s,z} Q^z \times \sum_{l=0}^{s-2z-1} \left(2\pi l \Phi^{r+s-2z-l,l}_{n+j-k-s+2z+l}(K) - Q \Phi^{r+s-2z-l,l-2}_{n+j-k-s+2z+l}(K) \right),$$

where the constants $\chi^{(1)}_{n,j,k,s,z}$ and $\chi^{(2)}_{n,j,k,s,z}$ are explicitly known.

The constants $\chi^{(1)}_{n,j,k,s,z}$ and $\chi^{(2)}_{n,j,k,s,z}$ only depend on the indicated lower indices. It is remarkable that they are independent of r. Moreover, the right-hand side also involves other tensor valuations than $\Phi^{r',s'}_{n-k+j}(K)$. For instance, in the special case where $n = 3$, $k = 2$, $j = 0$, $r = 1$ and $s = 2$, Theorem 2.2 yields that

$$\int_{A(3,2)} \Phi_0^{1,2}(K \cap E)\, \mu_2(\mathrm{d}E) = \frac{1}{3} \Phi_1^{1,2}(K) + \frac{1}{24\pi} Q \Phi_1^{1,0}(K) + \frac{1}{6} \Phi_0^{2,1}(K).$$

It can be shown that it is not possible to write $\Phi_0^{2,1}$ as a linear combination of $\Phi_1^{1,2}$ and $Q\Phi_1^{1,0}$, which are the only other Minkoski tensors of rank 3 and homogeneity degree 2.

The explicit expression obtained for the constants in [8] requires a multiple (five-fold) summation over products and ratios of binomial coefficients and Gamma functions. Some progress in simplifying this representation is described in Chap. **??**.

Since the tensor valuations on the right-hand side of this Crofton formula are not linearly independent, the specific representation is not unique. Using the linear relation due to McMullen, the result can also be expressed in the form

$$\int_{A(n,k)} \Phi_j^{r,s}(K \cap E)\, \mu_k(\mathrm{d}E) = \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \chi^{(1)}_{n,j,k,s,z} Q^z \Phi^{r,s-2z}_{n+j-k}(K) + \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor -1} \chi^{(2)}_{n,j,k,s,z} Q^z \times \sum_{l \geq s-2z} \left(Q \Phi^{r+s-2z-l,l-2}_{n+j-k-s+2z+l}(K) - 2\pi l \Phi^{r+s-2z-l,l}_{n+j-k-s+2z+l}(K) \right)$$

with the same constants as before. From this form, we now deduce the Crofton formula for the translation invariant tensor valuations Φ_j^s. For $r = 0$, the sum $\sum_{l \geq s-2z}$

on the right-hand side is non-zero only if $l = s - 2z$. Therefore, after some index shift (and discussion of the 'boundary cases' $z = 0$ and $z = \lfloor \frac{s}{2} \rfloor$), we obtain

$$\int_{A(n,k)} \Phi_j^s(K \cap E)\, \mu_k^n(\mathrm{d}E) = \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \chi_{n,j,k,s,z}^{(*)} Q^z \Phi_{n+j-k}^{s-2z}(K) \tag{1}$$

for $j < k$, where

$$\chi_{n,j,k,s,z}^{(*)} = \chi_{n,j,k,s,z}^{(1)} + \chi_{n,j,k,s,z-1}^{(2)} - 2\pi(s-2z)\chi_{n,j,k,s,z}^{(2)}.$$

Since the right-hand side is uniquely determined by the left-hand side and the tensor valuations on the right-hand side are linearly independent, the constant $\chi_{n,j,k,s,z}^{(*)}$ is uniquely determined. From the expression available for the right-hand side, it seems to be a formidable task to get a reasonably simple expression for this constant. If $j = k$, then Theorem 2.1 shows that (1) remains true if we define $\chi_{n,k,k,s,\lfloor \frac{s}{2} \rfloor} := \tilde{\alpha}_{n,k,s}$ if s is even, and as zero in all other cases. As we will see, the approach of algebraic integral geometry to (1) will reveal that $\chi_{n,j,k,s,z}^{(*)}$ is indeed a surprisingly simple expression.

To compare the algebraic approach with the one used in [8] and extended to tensorial curvature measures in Chap. **??**, we point out that the result of Theorem 2.2 is complemented by and in fact is based on an intrinsic Crofton formula, where the tensor valuation $\Phi_j^{r,s}(K \cap E)$ is replaced by $\Phi_{j,E}^{r,s}(K \cap E)$. The latter is the tensor valuation of the intersection $K \cap E$, but calculated with respect to E as the ambient space. The two tensors are connected by the relation (due to McMullen [10, Theorem 5.1], see also [8])

$$\Phi_j^{r,s}(K \cap E) = \sum_{m \geq 0} \frac{Q(E^\perp)^m}{(4\pi)^m m!} \Phi_{j,E}^{r,s-2m}(K \cap E), \tag{2}$$

where $Q(E^\perp)$ is the metric tensor of the linear subspace orthogonal to the direction space of E. Note that for $s = 0$ we get $\Phi_j^{r,0}(K \cap E) = \Phi_{j,E}^{r,0}(K \cap E)$, since the intrinsic volumes and the suitably normalized curvature measures are independent of the ambient space. The intrinsic Crofton formula for

$$\int_{A(n,k)} \Phi_{j,E}^{r,s}(K \cap E)\, \mu_k(\mathrm{d}E)$$

has the same structure as the extrinsic Crofton formula stated in Theorem 2.2, but the constants are different. Apart from reducing the number of summations required for determining the constants, progress in understanding the structure of these (intrinsic and extrinsic) integral geometric formulas can be made by localizing the Minkowski tensors. This is the topic of Chapter **??**.

Crofton and intersectional kinematic formulae for Minkowski tensors $\Phi_j^{r,s}$ with $s = 0$ are just special cases of corresponding (more general) integral geometric formulas for curvature measures. For example, we have

$$\int_{\mathbf{A}(n,k)} \Phi_j^{r,0}(K \cap E)\, \mu_k(\mathrm{d}E) = a_{njk} \cdot \Phi_{n+j-k}^{r,0}(K) \tag{3}$$

and

$$\int_{\mathbf{G}_n} \Phi_j^{r,0}(K \cap gM)\, \mu(\mathrm{d}g) = \sum_{k=j}^{n} a_{njk} \cdot \Phi_{n+j-k}^{r,0}(K) V_k(M), \tag{4}$$

where G_n is the Euclidean motion group, μ is the suitably normalized Haar measure and the (simple) constants a_{njk} are known explicitly. Therefore, we focus on the case $s \neq 0$ (and $r = 0$) in the following.

A close connection between Crofton formulae and intersectional kinematic formulae follows from Hadwiger's general integral geometric theorem (see [13, Theorem 5.1.2]). It states that for any continuous valuation φ on the space of convex bodies and for all $K, M \in \mathscr{K}^n$, we have

$$\int_{\mathbf{G}_n} \varphi(K \cap gM)\, \mu(\mathrm{d}g) = \sum_{k=0}^{n} \int_{\mathbf{A}(n,k)} \varphi(K \cap E)\, \mu_k(\mathrm{d}E)\, V_k(M). \tag{5}$$

Hence, if a Crofton formula for the functional φ is available, then an intersectional kinematic formula is an immediate consequence. This statement includes also tensor-valued functionals, since (5) can be applied coordinate-wise. In particular, this shows that (4) can be obtained from (3). In the same way, Theorem 2.2 and the special case shown in (1) imply kinematic formulas for intersections of convex bodies, one fixed the other moving. Thus, for instance, we obtain

$$\int_{\mathbf{G}_n} \Phi_j^s(K \cap gM)\, \mu(\mathrm{d}g) = \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{k=j}^{n} \chi_{n,j,k,s,z}^{(*)}\, Q^z \Phi_{n+j-k}^{s-2z}(K) V_k(M). \tag{6}$$

These results are related to and in fact inspired general integral geometric formulas for area measures (see [9]). The starting point is a local version of Hadwiger's general integral geometric theorem for measure valued valuations. To state it, let $\mathscr{M}^+(\mathbb{S}^{n-1})$ be the cone of non-negative measures in the vector space $\mathscr{M}(\mathbb{S}^{n-1})$ of finite Borel measures on the unit sphere. We write $\mathscr{K}'$ for the set of all non-empty convex bodies in $\mathscr{K}^n$.

Theorem 2.3. *Let $\varphi : \mathscr{K}' \to \mathscr{M}^+(\mathbb{S}^{n-1})$ be a continuous and additive mapping with $\varphi(\emptyset, \cdot) := 0$ (the zero measure). Then, for $K, M \in \mathscr{K}^n$ and Borel sets $A \subset \mathbb{S}^{n-1}$,*

$$\int_{G_n} \varphi(K \cap gM, A)\, \mu(dg) = \sum_{k=0}^{n} [T_{n,k}\varphi(K, \cdot)](A) V_k(M), \tag{7}$$

with (the Crofton operator) $T_{n,k} : \mathscr{M}^+(\mathbb{S}^{n-1}) \to \mathscr{M}^+(\mathbb{S}^{n-1})$ given by

$$T_{n,k}\varphi(K, \cdot) := \int_{A(n,k)} \varphi(K \cap E, \cdot)\, \mu_k(dE), \quad k = 0, \ldots, n.$$

Using the connection to mean section bodies and, for $p \in \{-1, 0, 1, \dots, n\}$, the Fourier operators I_p on smooth functions on the unit sphere, the following Crofton formula for area measures can be proved (see [9, Theorem 3.1]). Here I^* is the reflection operator $(I^* f)(u) = f(-u)$, $u \in \mathbb{S}^{n-1}$, for a smooth function f on the unit sphere.

Theorem 2.4. *Let For $1 \le j < q \le n$ and $K \in \mathscr{K}^n$. Then*

$$\int_{A(n,q)} S_j(K \cap E, \cdot)\, \mu_q(dE) = a(n, j, q) I_j I_{q-j} S_{n+j-q}(-K, \cdot) \tag{8}$$

with

$$a(n, j, q) = \frac{j}{2^n \pi^{(n+q)/2}(n + j - q)} \frac{\Gamma(\frac{q+1}{2})\Gamma(n - j)}{\Gamma(\frac{n+1}{2})\Gamma(q - j)}.$$

The operator $T_{n,j,q} := a(n, j, q) I_j I_{q-j} I^*$, for $1 \le j < q \le n$, and the identity operator $T_{n,j,n}$ act as linear operators on $\mathscr{M}(\mathbb{S}^{n-1})$ and can be used to express (8) in the form

$$\int_{A(n,q)} S_j(K \cap E, \cdot)\, \mu_q(dE) = T_{n,j,q} S_{n+j-q}(K, \cdot). \tag{9}$$

Combining equations (7) and (9), we obtain a kinematic formula for area measures. Using again the operator $T_{n,j,k}$, it can be stated in the form

$$\int_{G_n} S_j(K \cap gM, A)\, \mu(dg) = \sum_{k=j}^{d} [T_{n,j,k} S_{n+j-k}(K, \cdot)](A) V_k(M),$$

for $j = 1, \dots, n - 1$. Since the Fourier operators act as multiplier operators on spherical harmonics (see [9] for a summary of the main properties of this Fourier operator and for further references), it follows that Theorem 2.4 can be rewritten in the form

$$\begin{aligned} &\int_{A(n,q)} \int_{\mathbb{S}^{n-1}} f_s(u)\, S_j(K \cap E, du)\, \mu_q(dE) \\ &\qquad = a_s(n, j, q) \int_{\mathbb{S}^{n-1}} f_s(u)\, S_{n+j-q}(K, du), \end{aligned} \tag{10}$$

where f_s is a spherical harmonic of degree s and $a_s(s, j, q) = a(n, j, q) b_s(n, j, q)$ with

$$b_s(n, j, q) = 2^q \pi^n \frac{\Gamma\left(\frac{s+j}{2}\right)\Gamma\left(\frac{s+q-j}{2}\right)}{\Gamma\left(\frac{s+n-j}{2}\right)\Gamma\left(\frac{s+n-q+j}{2}\right)}.$$

In addition to Crofton and intersectional kinematic formulae, there is another type of integral geometric formula. Since they involve rotations and Minkowski sums of convex bodies, it is justified to call them rotation sum formulas. Let $\mathrm{SO}(n)$ denote the group of rotations and let ν denote the Haar probability measure on this group. A general form of such a formula can again be stated for area measures. Let

$K, M \in \mathscr{K}^n$ be convex bodies and let $\alpha, \beta \subset \mathbb{S}^{n-1}$ be Borel sets. Then we have (see [12, Theorem 4.4.6])

$$\int_{\mathrm{SO}(n)} \int_{\mathbb{S}^{n-1}} \mathbf{1}_\alpha(u)\mathbf{1}_\beta(\rho^{-1}u)\, S_j(K+\rho M, \mathrm{d}u)\, \nu(\mathrm{d}\rho) = \frac{1}{\omega_n} \sum_{k=0}^{j} \binom{j}{k} S_k(K,\alpha) S_{j-k}(M,\beta). \tag{11}$$

More generally, (11) (applied to coordinate functions) together with the inversion invariance of the Haar measure ν and basic measure theoretic extension arguments show that for any continuous function $f : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to \mathrm{Sym}^{s_1} \otimes \mathrm{Sym}^{s_2}$ we obtain

$$\int_{\mathrm{SO}(n)} \int_{\mathbb{S}^{n-1}} f(u, \rho u)\, S_j(K+\rho^{-1}M, \mathrm{d}u)\, \nu(\mathrm{d}\rho) = \frac{1}{\omega_n} \sum_{k=0}^{j} \binom{j}{k} \int_{(\mathbb{S}^{n-1})^2} f(u,v)\, \big(S_k(K,\cdot) \times S_{j-k}(M,\cdot)\big)\, (\mathrm{d}(u,v)).$$

If we now define (to simplify constants)

$$\phi_k^s(K) := \int_{\mathbb{S}^{n-1}} u^s\, S_k(K, \mathrm{d}u), \tag{12}$$

and choose $f(u,v) = u^{s_1} \otimes v^{s_2}$, then we get

$$\begin{aligned}
&\int_{\mathrm{SO}(n)} (\mathrm{id}^{\otimes s_1} \otimes \rho^{\otimes s_2}) \phi_j^{s_1+s_2}(K+\rho^{-1}M)\, \nu(\mathrm{d}\rho) \\
&\quad = \int_{\mathrm{SO}(n)} \int_{\mathbb{S}^{n-1}} u^{s_1} \otimes (\rho u)^{s_2}\, S_j(K+\rho^{-1}M, \mathrm{d}u)\, \nu(\mathrm{d}\rho) \\
&\quad = \frac{1}{\omega_n} \sum_{k=0}^{j} \binom{j}{k} \int_{(\mathbb{S}^{n-1})^2} u^{s_1} \otimes v^{s_2}\, \big(S_k(K,\cdot) \times S_{j-k}(M,\cdot)\big)\, (\mathrm{d}(u,v)),
\end{aligned} \tag{13}$$

and hence

$$\int_{\mathrm{SO}(n)} (\mathrm{id}^{\otimes s_1} \otimes \rho^{\otimes s_2}) \phi_j^{s_1+s_2}(K+\rho^{-1}M)\, \nu(\mathrm{d}\rho) = \frac{1}{\omega_n} \sum_{k=0}^{j} \binom{j}{k} \phi_k^{s_1}(K) \otimes \phi_{j-k}^{s_2}(M).$$

Up to the different normalization, this is the additive kinematic formula for tensor valuations stated in [7, Theorem 5]. In particular,

$$\int_{\mathrm{SO}(n)} \phi_j^s(K+\rho M)\, \nu(\mathrm{d}\rho) = \frac{1}{\omega_n} \sum_{k=0}^{j} \binom{j}{k} \phi_k^s(K) S_{j-k}(M),$$

where $S_i(M) := S_i(M, \mathbb{S}^{n-1}) = n\kappa_{n-i}\binom{n}{i}^{-1} V_i(M)$.

In the following section, we develop basic algebraic structures for tensor valuations and provide applications to integral geometry. From this approach, we will obtain a Crofton formula for the tensor valuations Φ_k^s, but also for another set of tensor valuations, denoted by Ψ_k^s, for which the Crofton formula has 'diagonal form'. Moreover, we will study more general intersectional kinematic formulae than the one considered in (6) and describe the connection between intersectional and additive kinematic formulae. In the course of our analysis, we determine Alesker's Fourier operator for spherical valuations, that is, valuations obtained by integration of a spherical harmonic (or, more generally, any smooth spherical function) against an area measure.

2 Algebraic Structures on Tensor Valuations

We let $\mathrm{Val} = \mathrm{Val}(\mathbb{R}^n)$ denote the Banach space of translation invariant continuous valuations on $V = \mathbb{R}^n$, and let $\mathrm{Val}^\infty = \mathrm{Val}^\infty(\mathbb{R}^n)$ be the dense subspace of smooth valuations, see [5] and Chap. 4 for more information. In this section, we first discuss the extension of basic operations and transformations from scalar valuations to tensor-valued valuations. The scalar case is described in Chap. 4.

2.1 Product

Existence and uniqueness of the product of smooth valuations is provided by the following result.

Proposition 2.5. *Let $\phi_1, \phi_2 \in \mathrm{Val}^\infty$ be smooth valuations on $\mathbb{R}^n$ given by*

$$\phi_i(K) = \mathrm{vol}(K + A_i), \quad K \in \mathscr{K}^n,$$

where $A_1, A_2 \in \mathscr{K}^n$ are smooth convex bodies with positive Gauss curvature at every boundary point. Let $\Delta : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ be the diagonal embedding. Then

$$\phi_1 \cdot \phi_2(K) := \mathrm{vol}(\Delta K + A_1 \times A_2), \quad K \in \mathscr{K}^n,$$

extends by continuity and bilinearity to a product on Val^∞.

The product is compatible with the degree of a valuation (i.e., if ϕ_i has degree k_i, then $\phi_1 \cdot \phi_2$ has degree $k_1 + k_2$ if $k_1 + k_2 \le n$), and more generally with the action of the group $\mathrm{GL}(n)$.

We can extend the product component-wise from scalar-valued smooth valuations to smooth tensor valued valuations. If $\Phi_1(K) = \sum_{i=1}^m \phi_i(K) w_i$, where $w_1, \ldots, w_m$ is a basis of $\mathrm{Sym}^{s_1} V$, and $\Phi_2(K) = \sum_{j=1}^l \psi_j(K) u_j$, where $u_1, \ldots, u_l$ is a basis of $\mathrm{Sym}^{s_2} V$, then

$$(\Phi_1 \cdot \Phi_2)(K) = \sum_{i,j} (\phi_i \cdot \psi_j)(K) w_i u_j.$$

The dot on the right-hand side is the product of smooth valuations and $w_i u_j \in \mathrm{Sym}^{s_1+s_2} V$ denotes the symmetric tensor product of the symmetric tensors $w_i \in \mathrm{Sym}^{s_1} V$ and $u_j \in \mathrm{Sym}^{s_2} V$.

Let us verify that this definition is independent of the chosen bases. Let $w_i' = \sum_j c_{ij} w_j$ with some invertible matrix (c_{ij}), and let $u_i' = \sum_j e_{ij} u_j$ with an invertible matrix (e_{ij}).

If $\Phi_1(K) = \sum_i \phi_i'(K) w_i' = \sum_i \phi_i(K) w_i$, then a comparison of coefficients yields $\phi_i' = \sum c^{ji} \phi_j$, where (c^{ij}) is the matrix inverse. Similarly, from $\Phi_2(K) = \sum_i \psi_j'(K) u_i' = \sum_i \psi_i(K) u_i$, then $\psi_i' = \sum e^{ji} \psi_j$. Therefore

$$\begin{aligned}
\sum_{i,j} (\phi_i' \cdot \psi_j') w_i' u_j' &= \sum_{i,j,b_1,b_2} \left(\sum_{a_1,b_1} c^{a_1 i} \phi_{a_1} \cdot e^{b_1 j} \psi_{b_1} \right) \sum_{a_2,b_2} c_{ia_2} w_{a_2} e_{jb_2} u_{b_2} \\
&= \sum_{a_1,a_2,b_1,b_2} \underbrace{\left(\sum_{i,j} c^{a_1 i} c_{ia_2} e^{b_1 j} e_{jb_2} \right)}_{=\delta_{a_2}^{a_1} \delta_{b_2}^{b_1}} (\phi_{a_1} \cdot \psi_{b_1}) w_{a_2} \cdot u_{b_2} \\
&= \sum_{a,b} (\phi_a \cdot \psi_a) w_a \cdot u_b,
\end{aligned}$$

which proves the asserted independence of the representation. Writing $\mathrm{TVal}^s(V)$ for the vector space of translation invariant continuous valuations on $\mathscr{K}(V)$ with values in the vector space $\mathrm{Sym}^s V$ of symmetric tensors of rank s over V, and $\mathrm{TVal}^{s,\infty}(V)$ for the smooth elements of this vector space, we have

$$\Phi_1 \cdot \Phi_2 \in \mathrm{TVal}_{k+l}^{s_1+s_2,\infty}(V), \quad k+l \leq n,$$

for $\Phi_1 \in \mathrm{TVal}_k^{s_1,\infty}(V)$, $\Phi_2 \in \mathrm{TVal}_l^{s_2,\infty}(V)$ and $k,l \in \{0,\ldots,n\}$.

A similar description and similar arguments can be given for the operation considered in the following subsection.

2.2 Convolution

Similarly as for the product of valuations, an explicit definition of the convolution of two valuations is given only for a suitable subclass of valuations.

Proposition 2.6. *Let $\phi_1, \phi_2 \in \mathrm{Val}^\infty$ be smooth valuations on $\mathbb{R}^n$ given by*

$$\phi_i(K) = \mathrm{vol}(K + A_i),$$

where A_1, A_2 are smooth convex bodies with positive Gauss curvature at every boundary point. Then

$$\phi_1 * \phi_2(K) := \mathrm{vol}(K + A_1 + A_2),$$

extends by continuity and bilinearity to a product (which is called convolution) on Val^∞.

Written in invariant terms, the convolution is a bilinear map

$$\mathrm{Val}^{sm}(V)\otimes \mathrm{Dens}(V^*)\times \mathrm{Val}^{sm}(V)\otimes \mathrm{Dens}(V^*)\to \mathrm{Val}^{sm}(V)\otimes \mathrm{Dens}(V^*).$$

It is compatible with the action of the group $\mathrm{GL}(n)$ and with the codegree of a valuation (i.e., if ϕ_i has degree k_i, then $\phi_1 * \phi_2$ has degree k_1+k_2-n if $k_1+k_2\geq n$).

The convolution can be extended component-wise to a convolution on the space of translation invariant smooth tensor valuations. Hence we have

$$\Phi_1 * \Phi_2 \in \mathrm{TVal}_{k+l-n}^{s_1+s_2,\infty}(V),\quad k+l\geq n,$$

for $\Phi_1\in \mathrm{TVal}_k^{s_1,\infty}(V)$, $\Phi_2\in \mathrm{TVal}_l^{s_2,\infty}(V)$ and $k,l\in\{0,\dots,n\}$. This is analogous to the definition and computation in the previous subsection.

2.3 Alesker-Fourier Transform

Alesker introduced an operation on smooth valuations, called Alesker-Fourier transform. It is a map

$$\mathbb{F}:\mathrm{Val}_k^\infty(\mathbb{R}^n)\to \mathrm{Val}_{n-k}^\infty(\mathbb{R}^n)$$

which satisfies

$$\mathbb{F}(\phi_1\cdot\phi_2)=\mathbb{F}\phi_1 * \mathbb{F}\phi_2. \tag{14}$$

On valuations which are smooth and even, it can easily be described in terms of Klain functions as follows. Let $\phi\in \mathrm{Val}_k^{\infty,+}(\mathbb{R}^n)$ (the space of smooth and even valuations which are homogeneous of degree k). Then the restriction of ϕ to a k-dimensional subspace E is a multiple $\mathrm{Kl}_\phi(E)$ of the volume, and the resulting function (Klain function) Kl_ϕ determines ϕ. Then

$$\mathrm{Kl}_{\mathbb{F}\phi}(E)=\mathrm{Kl}_\phi(E^\perp)$$

for every $(n-k)$-dimensional subspace E.

As an example, the intrinsic volumes satisfy

$$\mathbb{F}(\mu_k)=\mu_{n-k}. \tag{15}$$

The description in the odd case is more involved and better to understand in invariant terms (i.e., without referring to a Euclidean structure).

Let V be an n-dimensional real vector space. Then

$$\mathbb{F}:\mathrm{Val}_k^\infty(V)\to \mathrm{Val}_{n-k}^\infty(V)\otimes \mathrm{Dens}(V^*),$$

where Dens denotes the one-dimensional space of densities. This map commutes with the action of $\mathrm{GL}(V)$ on both sides. Applying it twice (and using the identification $\mathrm{Dens}(V^*) \otimes \mathrm{Dens}(V) \cong \mathbb{C}$), it satisfies the Plancherel type formula

$$\mathbb{F}^2 \phi(K) = \phi(-K).$$

Working again on Euclidean space $V = \mathbb{R}^n$, we can extend the Alesker-Fourier transform component-wise to a map

$$\mathbb{F} : \mathrm{TVal}_k^{s,\infty} \to \mathrm{TVal}_{n-k}^{s,\infty}.$$

It is not an easy task to determine the Fourier transform of valuations other than the intrinsic volumes.

2.4 Example: Intrinsic Volumes

As an example, let us compute the Alesker product of intrinsic volumes.

Recall Steiner's formula which states that

$$\mathrm{vol}(K + rB) = \sum_{i=0}^{n} \mu_{n-i}(K)\kappa_i r^i, \qquad r \geq 0.$$

Now fix r and s and define the valuations $\phi_1(K) := \mathrm{vol}(K + rB)$ and $\phi_2(K) := \mathrm{vol}(K + sB)$. Then

$$\phi_1 * \phi_2(K) = \mathrm{vol}(K + rB + sB) = \mathrm{vol}(K + (r+s)B) = \sum_{k=0}^{n} \mu_{n-k}(K)\kappa_k (r+s)^k,$$

hence

$$\phi_1 * \phi_2 = \sum_{i,j=0}^{n} \mu_{n-i-j}\kappa_{i+j}\binom{i+j}{i} r^i s^j.$$

On the other hand, since $\phi_1 = \sum_{i=0}^{n} \mu_{n-i}\kappa_i r^i$ and $\phi_2 = \sum_{i=0}^{n} \mu_{n-i}\kappa_i s^i$, we obtain

$$\phi_1 * \phi_2 = \sum_{i,j=0}^{n} \mu_{n-i} * \mu_{n-j}\kappa_i\kappa_j r^i s^j.$$

Now we compare the coefficient of $r^i s^j$ in these equations and get

$$\mu_{n-i-j}\kappa_{i+j}\binom{i+j}{i} = \mu_{n-i} * \mu_{n-j}\kappa_i\kappa_j.$$

Writing i instead of $n-i$ and j instead of $n-j$, we obtain

$$\mu_i * \mu_j = \begin{bmatrix} 2n-i-j \\ n-i \end{bmatrix} \mu_{i+j-n}, \tag{16}$$

where we used the flag coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} := \binom{n}{k} \frac{\kappa_n}{\kappa_k \kappa_{n-k}}, \qquad k \in \{0, \dots, n\}.$$

Taking Alesker-Fourier transform on both sides yields

$$\mu_i \cdot \mu_j = \begin{bmatrix} i+j \\ i \end{bmatrix} \mu_{i+j}. \tag{17}$$

The computation of convolution and product of tensor valuations follows the same scheme: first one computes the convolution of tensor valuations, which can be considered easier. Then one applies the Alesker-Fourier transform to obtain the product. However, in the tensor-valued case it is much harder to write down the Alesker-Fourier transform in an explicit way. This step is the technical heart of our approach.

2.5 *Poincaré Duality*

The product of smooth translation invariant valuations as well as the convolution both satisfy a version of Poincaré duality, which moreover are identical up to a sign.

Recall that $\mathrm{Val}_0 \cong \mathbb{R} \cdot \chi$, $\mathrm{Val}_n \cong \mathbb{R} \cdot \mathrm{vol}$, where vol is any choice of Lebesgue measure. We denote by $\phi_0, \phi_n \in \mathbb{R}$ the component of $\phi \in \mathrm{Val}$ of degree 0 and n respectively.

Proposition 2.7. *The pairings*

$$\mathrm{Val}_k^\infty \times \mathrm{Val}_{n-k}^\infty \to \mathbb{R}, \quad (\phi_1, \phi_2) \mapsto (\phi_1 \cdot \phi_2)_n,$$

and

$$\mathrm{Val}_k^\infty \times \mathrm{Val}_{n-k}^\infty \to \mathbb{R}, \quad (\phi_1, \phi_2) \mapsto (\phi_1 * \phi_2)_0,$$

are perfect, that is, the induced maps

$$\mathrm{pd}_m, \mathrm{pd}_c : \mathrm{Val}_k^\infty \to \mathrm{Val}_{n-k}^{\infty,*}$$

are injective with dense image. Moreover,

$$\mathrm{pd}_c = \begin{cases} \mathrm{pd}_m & \text{on } \mathrm{Val}_k^+ . \\ -\mathrm{pd}_m & \text{on } \mathrm{Val}_k^- . \end{cases}$$

To illustrate this proposition and to highlight the difference between the two pairings, let us compute them on an easy example. Let $\phi_i(K) := \mathrm{vol}(K + A_i)$, where A_i, $i \in \{1,2\}$, are smooth convex bodies with positive Gauss curvature. Then $\phi_1 * \phi_2(K) = \mathrm{vol}(K + A_1 + A_2)$, and hence $(\phi_1 * \phi_2)_0 = \mathrm{vol}(A_1 + A_2)$.

On the other hand, $\phi_1 \cdot \phi_2(K) = \mathrm{vol}_{2n}(\Delta K + A_1 \times A_2)$. Using Fubini's theorem, one rewrites this as

$$\phi_1 \cdot \phi_2(K) = \int_{\mathbb{R}^n} \phi_2((x - A_1) \cap K)\,\mathrm{d}x.$$

Taking K a large ball reveals that $(\phi_1 \cdot \phi_2)_n = \phi_2(-A_1) = \mathrm{vol}(A_2 - A_1)$. If $A_1 = -A_1$, then ϕ_1 is even and both pairings agree indeed.

2.6 *Explicit Computation in the* $\mathrm{O}(n)$*-Equivariant Case*

In this subsection, we outline the explicit computation of product, convolution and Alesker-Fourier transform in the $\mathrm{O}(n)$-equivariant case. Depending on the situation, we will either use the basis consisting of the tensor valuations $Q^i \Phi_k^{s-2i}$ or the basis consisting of the tensor valuations $Q^i \Psi_k^{s-2i}$. The latter are defined in the following proposition.

Proposition 2.8. *The following statements hold.*

(i) *For* $0 \leq k < n$ *and* $s \neq 1$, *define*

$$\Psi_k^s := \Phi_k^s + \sum_{j=1}^{\lfloor \frac{s}{2} \rfloor} \frac{(-1)^j \Gamma(\frac{n-k+s}{2}) \Gamma(\frac{n}{2} + s - 1 - j)}{(4\pi)^j j! \Gamma(\frac{n-k+s}{2} - j) \Gamma(\frac{n}{2} + s - 1)} Q^j \Phi_k^{s-2j}$$

and let $\Psi_n^0 := \Phi_n^0$. *Then* Ψ_k^s *is the trace free part of* Φ_k^s. *In particular,* $\Psi_k^s \equiv \Phi_k^s \mod Q$.

(ii) *For* $0 \leq k < n$ *and* $s \neq 1$, Φ_k^s *can be written in terms of* $\Psi_k^{s'}$ *as*

$$\Phi_k^s = \Psi_k^s + \sum_{j=1}^{\lfloor \frac{s}{2} \rfloor} \frac{\Gamma\left(\frac{n-k+s}{2}\right) \Gamma(\frac{n}{2} + s - 2j)}{(4\pi)^j j! \Gamma(\frac{n-k+s}{2} - j) \Gamma(\frac{n}{2} + s - j)} Q^j \Psi_k^{s-2j}.$$

The inversion which is needed to derive (ii) from (i) can be accomplished with the help of Zeilberger's algorithm.

The first and easier step is to compute the convolution of two tensor valuations. Since Φ_k^s is smooth (i.e., each component is a smooth valuation), we may write

$$\Phi_k^s(K) = \int_{\mathrm{nc}(K)} \omega_{k,s},$$

where $\omega_{k,s}$ is a smooth $(n-1)$-form on the sphere bundle $\mathbb{R}^n \times S^{n-1}$ with values in $\mathrm{Sym}^s \mathbb{R}^n$. Next, for valuations represented by differential forms, there is an easy formula for the convolution, which involves only some linear and bilinear operations (a kind of Hodge star and a wedge product). The resulting formula is that for $k,l \leq n$ with $k+l \geq n$ and $s_1,s_2 \neq 1$, we have

$$\begin{aligned}\Phi_k^{s_1} * \Phi_l^{s_2} = \frac{\omega_{s_1+s_2+2n-k-l}}{\omega_{s_1+n-k}\omega_{s_2+n-l}} \frac{(n-k)(n-l)}{2n-k-l} \cdot \\ \cdot \binom{2n-k-l}{n-k}\binom{s_1+s_2}{s_1}\frac{(s_1-1)(s_2-1)}{1-s_1-s_2}\Phi_{k+l-n}^{s_1+s_2},\end{aligned}$$

or, using the normalization (12) which is more convenient for this purpose,

$$\phi_k^{s_1} * \phi_l^{s_2} = n\frac{\binom{k+l}{n}}{\binom{k+l}{k}}\frac{(s_1-1)(s_2-1)}{1-s_1-s_2}\phi_{k+l-n}^{s_1+s_2}.$$

The computation of the Alesker-Fourier transform of tensor valuations is the main step and will be explained in the next subsection. For $0 \leq k \leq n$ and $s \neq 1$, the result is

$$\begin{aligned}\mathbb{F}(\Psi_k^s) &= \mathbf{i}^s\, \Psi_{n-k}^s, \\ \mathbb{F}(\Phi_k^s) &= \mathbf{i}^s \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \frac{(-1)^j}{(4\pi)^j j!} Q^j \Phi_{n-k}^{s-2j}.\end{aligned}$$

Finally, the product of two tensor valuations can be computed once the convolution and the Alesker-Fourier transform are known, see (14). The result is a bit more involved than the formulas for convolution and Alesker-Fourier transform. The reason is that the formula for convolution is best described in terms of the tensor valuations Φ_k^s, while the description of the Alesker-Fourier transform has a simpler expression for the Ψ_k^s.

After some algebraic manipulations (which make use of Zeilberger's algorithm), we arrive at

$$\begin{aligned}\Phi_k^{s_1} \cdot \Phi_l^{s_2} = \frac{kl}{k+l}\binom{k+l}{k} \sum_{\substack{a=0 \\ 2a \neq s_1+s_2-1}}^{\lfloor \frac{s_1+s_2}{2} \rfloor} \frac{1}{(4\pi)^a a!}\Bigg(\sum_{m=0}^{a} \sum_{i=\max\{0,m-\lfloor \frac{s_2}{2} \rfloor\}}^{\min\{m,\lfloor \frac{s_1}{2} \rfloor\}} \\ (-1)^{a-m}\binom{a}{m}\binom{m}{i}\frac{\omega_{s_1+s_2-2m+k+l}}{\omega_{s_1-2i+k}\omega_{s_2-2m+2i+l}}\binom{s_1+s_2-2m}{s_1-2i} \cdot \\ \cdot \frac{(s_1-2i-1)(s_2-2m+2i-1)}{1-s_1-s_2+2m}\Bigg) Q^a \Phi_{k+l}^{s_1+s_2-2a}. \qquad (18)\end{aligned}$$

Here $0 \leq k,l$ with $k+l \leq n$ and $s_1,s_2 \neq 1$. It seems that there is no closed expression for the inner sum.

2.7 Tensor Valuations Versus Scalar-Valued Valuations

The interplay between tensor valuations and scalar valued valuations will be essential in the computation of the Alesker-Fourier transform. We therefore explain this now is some more detail.

We first need some facts from representation theory. It is well-known that equivalence classes of complex irreducible (finite-dimensional) representations of $\mathrm{SO}(n)$ are indexed by their highest weights. The possible highest weights are tuples $(\lambda_1, \lambda_2, \ldots, \lambda_{\lfloor \frac{n}{2} \rfloor})$ of integers such that

1. $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\lfloor \frac{n}{2} \rfloor} \geq 0$ if n is odd,
2. $\lambda_1 \geq \lambda_2 \geq \ldots \geq |\lambda_{\frac{n}{2}}| \geq 0$ if n is even.

Given $\lambda = (\lambda_1, \ldots, \lambda_{\lfloor \frac{n}{2} \rfloor})$ satisfying this condition, we will denote the corresponding equivalence class of representations by Γ_λ.

The decomposition of the $\mathrm{SO}(n)$-module Val_k has recently been obtained in [3].

Theorem 2.9 ([3]). *There is an isomorphism of* $\mathrm{SO}(n)$*-modules*

$$\mathrm{Val}_k \cong \bigoplus_\lambda \Gamma_\lambda,$$

where λ *ranges over all highest weights such that* $|\lambda_2| \leq 2$*,* $|\lambda_i| \neq 1$ *for all* i *and* $\lambda_i = 0$ *for* $i > \min\{k, n-k\}$*. In particular, these decompositions are multiplicity-free.*

Let Γ be an irreducible representation of $\mathrm{SO}(n)$ and Γ^* its dual. The space of k-homogeneous $\mathrm{SO}(n)$-equivariant Γ-valued valuations (i.e., maps $\Phi : \mathscr{K} \to \Gamma$ such that $\Phi(gK) = g\Phi(K)$ for all $g \in \mathrm{SO}(n)$) is $(\mathrm{Val}_k \otimes \Gamma)^{\mathrm{SO}(n)} = \mathrm{Hom}_{\mathrm{SO}(n)}(\mathrm{Val}_k, \Gamma^*)$. By Theorem (2.9), Γ^* appears in the decomposition of Val_k precisely if Γ appears, and in this case the multiplicity is 1. By Schur's lemma it follows that $\dim(\mathrm{Val}_k \otimes \Gamma)^{\mathrm{SO}(n)} = 1$ in this case.

Let us construct the (unique up to scale) equivariant Γ-valued valuation explicitly. Denote by $\mathrm{Val}_k \cap \Gamma$ the Γ-isotypical component, which is isomorphic to Γ since Val_k is multiplicity free.

Let $\phi_1, \ldots, \phi_m$ be a basis of $\mathrm{Val}_k \cap \Gamma$. These elements play two different roles: first we can look at them as valuations, i.e., elements of Val_k. Second, we may think of $\phi_1, \ldots, \phi_m$ as basis of the irreducible representation Γ. The action of $\mathrm{SO}(n)$ on this basis is given by

$$g\phi_i = \sum_j c_i^j(g)\phi_j,$$

where $(c_i^j(g))_{i,j}$ is a matrix depending on g. The map $g \mapsto (c_i^j(g))_{i,j}$ is a homomorphism of Lie groups $\mathrm{SO}(n) \to \mathrm{GL}(m)$.

Let $\phi_1^*, \ldots, \phi_m^*$ be the dual basis of Γ^*. Then

$$g\phi_i^* = \sum_j (c_i^j(g))^{-t}\phi_j = \sum_j c_j^i(g^{-1})\phi_j,$$

Using the double role played by the ϕ_i mentioned above, we set

$$\Phi(K) := \sum_i \phi_i(K)\phi_i^* \in \Gamma^*. \tag{19}$$

We claim that Φ is $\mathrm{O}(n)$-equivariant valuation with values in Γ^*. Indeed, we compute

$$\begin{aligned}
\Phi(gK) &= \sum_i \phi_i(gK)\phi_i^* \\
&= \sum_i (g^{-1}\phi_i)(K)\phi_i^* \\
&= \sum_{i,j} c_i^j(g^{-1})\phi_j(K)\phi_i^* \\
&= \sum_j \phi_j(K) \sum_i c_i^j(g^{-1})\phi_i^* \\
&= \sum_j \phi_j(K) g\phi_j^* \\
&= g(\Phi(K)).
\end{aligned}$$

Conversely, start with an equivariant Γ^*-valued continuous translation invariant valuation Φ of degree k. Let $w_1, \dots, w_m$ be a basis of Γ^*. Then we may look at the components of Φ, i.e., we decompose

$$\Phi(K) = \sum_i \phi_i(K) w_i$$

with $\phi_i \in \mathrm{Val}_k$. Let the action of $\mathrm{SO}(n)$ on Γ^* be given by

$$g w_i = \sum_j a_i^j(g) w_j.$$

We have

$$\begin{aligned}
\Phi(gK) &= \sum_i \phi_i(gK) w_i = \sum_i (g^{-1}\phi_i)(K) w_i \\
g(\Phi(K)) &= \sum_j \phi_j(K) g w_j = \sum_{i,j} \phi_j(K) a_j^i(g) w_i.
\end{aligned}$$

Comparing coefficient yields $g^{-1}\phi_i = \sum a_j^i(g)\phi_j$, or

$$g\phi_i = \sum a_j^i(g^{-1})\phi_j.$$

This shows that the subspace of Val_k spanned by $\phi_1, \dots, \phi_m$ is isomorphic to Γ.

In summary, we have shown the following fact.

Each $\mathrm{SO}(n)$*-irreducible representation* Γ *appearing in the decomposition of* Val_k *corresponds to the (unique up to scale)* Γ^**-valued continuous translation invari-*

ant valuation Φ from (19). *Conversely, the coefficients of a Γ^*-valued continuous translation invariant valuation span a subspace of* Val_k *isomorphic to Γ.*

Let us now discuss the special case of symmetric tensor valuations. The $\mathrm{SO}(n)$-representation space Sym^s is (in general) not irreducible. Indeed, the trace map $\mathrm{tr} : \mathrm{Sym}^s \to \mathrm{Sym}^{s-2}$ commutes with $\mathrm{SO}(n)$, hence its kernel is an invariant subspace. This subspace turns out to be the irreducible representation $\Gamma_{(s,0,\dots,0)}$ and can be identified with the space $\mathscr{H}_s^n$ of harmonic polynomials of degree s. Recall that a homogeneous polynomial p of degree s on $\mathbb{R}^n$ is called harmonic if $\Delta p = 0$, where Δ is the Laplace operator. We refer to [12] for more information on spherical harmonics.

Since the trace map is onto, we get the following decomposition.

$$\mathrm{Sym}^s \cong \bigoplus_j \mathscr{H}^n_{s-2j}.$$

Instead of studying Sym^s-valued valuations, we can therefore study $\mathscr{H}_s^n$-valued valuations. For $s \neq 1$ and $1 \leq k \leq n-1$, the representation $\mathscr{H}_s^n$ appears in Val_k with multiplicity 1. Since $\mathscr{H}_s^n$ is self-dual, the construction sketched above yields in the special case $\Gamma := \mathscr{H}_s^n$ a unique (up to scale) $\mathscr{H}_s^n$-valued equivariant continuous translation invariant valuation of degree k, which was denoted by $\Psi_{k,s}$ in the introduction.

2.8 The Alesker-Fourier Transform

As we have seen in the previous subsection, the study of (symmetric) tensor valuations and the study of the $\mathscr{H}^s$-isotypical component of Val_k is equivalent. For the computation of the Alesker-Fourier transform, it is easier to work with scalar-valued valuations. Let us first define some particular class of valuations, called spherical valuations.

Let f be a smooth function on S^{n-1}. For $k \in \{0,\dots,n-1\}$, we define a valuation $\mu_{k,f} \in \mathrm{Val}_k(\mathbb{R}^n)$ by

$$\mu_{k,f}(K) := \binom{n-1}{k} \frac{1}{\omega_{n-k}} \int_{S^{n-1}} f(y)\, S_k(K, \mathrm{d}y).$$

Such valuations are called spherical (see also the recent preprint [14]). By Subsection 2.7, the components of an $\mathrm{SO}(n)$-equivariant tensor valuation are spherical. Since the Alesker-Fourier transform of such a tensor valuation is defined component-wise, it suffices to compute the Alesker-Fourier transform of spherical valuations.

In this subsection, we sketch this (rather involved) computation. The first and easy observation is that, by Schur's lemma, there exist constants $c_{n,k,s} \in \mathbb{C}$ which only depend on n,k,s such that

$$\mathbb{F}(\mu_{k,f}) = c_{n,k,s}\mu_{n-k,f}, \qquad f \in \mathscr{H}_s^n. \tag{20}$$

These multipliers of the Alesker-Fourier transform can be computed in the even case (i.e., s is even) by looking at Klain functions. In the odd case, there seems to be no easy way to compute them. We adapt ideas from [11], where the multipliers of the α-cosine transform was computed, to our situation. The main point is that the Alesker-Fourier transform is not only an $\mathrm{SO}(n)$-equivariant operator, but (if written in intrinsic terms) is equivariant under the larger group $\mathrm{GL}(n)$. Using elements from the Lie algebra $\mathfrak{gl}(n)$ allows us to pass from one irreducible $\mathrm{SO}(n)$-representation to another and to obtain a recursive formula for the constants $c_{n,k,s}$, which states that

$$\frac{c_{n,k,s+2}}{c_{n,k,s}} = -\frac{k+s}{n-k+s}. \tag{21}$$

This step requires extensive computations using differential forms, and we refer to [7] for the details.

Next, one can use induction over s, k, n to prove that

$$c_{n,k,s} = \mathbf{i}^s \frac{\Gamma\left(\frac{n-k}{2}\right)\Gamma\left(\frac{s+k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{s+n-k}{2}\right)}.$$

More precisely, in the even case, we may use as induction start the case $s = 0$, which corresponds to intrinsic volumes, whose Alesker-Fourier transform is known by (15).

In the odd case, we use as induction start $s = 3$. In order to compute $c_{n,k,3}$, we use the Crofton formula from [8] to compute the quotients $\frac{c_{n,k+1,3}}{c_{n,k,3}}$. This fixes all constants up to a scaling which may depend on n. More precisely,

$$c_{n,k,s} = \varepsilon_n \mathbf{i}^s \frac{\Gamma\left(\frac{n-k}{2}\right)\Gamma\left(\frac{s+k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{s+n-k}{2}\right)}, \tag{22}$$

where ε_n depends only on n. Using functorial properties of the Alesker-Fourier transform, we find that ε_n is independent of n. In the two-dimensional case, however, there is a very explicit description of the Alesker-Fourier transform which finally allows us to deduce that $\varepsilon_n = 1$ for all $n \geq 2$.

An alternative approach to determining the constants $c_{n,k,s}$ is to prove independently a Crofton formula for the tensor valuations Ψ_k^s, via the Crofton formula for area measures, as described before (see also Remark 4.6 in [9]). This point of view suggests to relate the Fourier operator for spherical valuations to the Fourier operators for spherical functions via the relation

$$\mathbb{F}(\bar{\mu}_{k,f}) = (2\pi)^{-\frac{d}{2}}\, \bar{\mu}_{d-k,I_k f},$$

for $f \in C^\infty(S^{d-1})$, where

$$\bar{\mu}_{k,f}(K) = \binom{d-1}{k}(2\pi)^{\frac{k}{2}} \int_{S^{d-1}} f(u)\, S_k(K, \mathrm{d}u),$$

is just a renormalization of $\mu_{k,f}(K)$.

3 Kinematic Formulas

In this section, we first describe the interplay between algebraic structures and kinematic formulas in general (i.e., for tensor valuations which are equivariant under a group G acting transitively on the unit sphere). Then we will specialize to the $\mathrm{O}(n)$-covariant case.

3.1 Relation Between Kinematic Formulas and Algebraic Structures

Let G be a subgroup of $\mathrm{O}(n)$ which acts transitively on the unit sphere. Then the space $\mathrm{TVal}^G(V)$ of G-covariant, translation invariant continuous tensor-valued valuations is finite-dimensional. Next we define two integral geometric operators. We start with the one for rotation sum formulas.

Let $\Phi \in \mathrm{TVal}^{s_1+s_2,G}(V)$. We define a bivaluation with values in the tensor product $\mathrm{Sym}^{s_1} V \otimes \mathrm{Sym}^{s_2} V$ by

$$a^G_{s_1,s_2}(\Phi)(K,L) := \int_G (\mathrm{id}^{\otimes s_1} \otimes g^{\otimes s_2})\Phi(K + g^{-1}L)\, \nu(\mathrm{d}g)$$

for $K, L \in \mathscr{K}(V)$, where G is endowed with the Haar probability measure ν (see [15]). (This notation is consistent with the case $V = \mathbb{R}^n$ and $G = \mathrm{O}(n)$.)

Let $\Phi_1, \dots, \Phi_{m_1}$ be a basis of $\mathrm{TVal}^{s_1,G}(V)$, and let $\Psi_1, \dots, \Psi_{m_2}$ be a basis of $\mathrm{TVal}^{s_2,G}(V)$. Arguing as in the classical Hadwiger argument (cf. [15, Theorem 4.3]), it can be seen that there are constants c^{Φ}_{ij} such that

$$a^G_{s_1,s_2}(\Phi)(K,L) = \sum_{i,j} c^{\Phi}_{ij}\, \Phi_i(K) \otimes \Psi_j(L)$$

for $K, L \in \mathscr{K}(V)$. The *additive kinematic operator* is the map

$$\begin{aligned} a^G_{s_1,s_2} : \mathrm{TVal}^{s_1+s_2,G}(V) &\to \mathrm{TVal}^{s_1,G}(V) \otimes \mathrm{TVal}^{s_2,G}(V) \\ \Phi &\mapsto \sum_{i,j} c^{\Phi}_{ij}\, \Phi_i \otimes \Psi_j, \end{aligned}$$

which is independent of the choice of the bases.

In view of intersectional kinematic formulas, we define a bivaluation with values in $\mathrm{Sym}^{s_1} V \otimes \mathrm{Sym}^{s_2} V$ by

$$k^G_{s_1,s_2}(\Phi)(K,L) := \int_{\bar{G}} (\mathrm{id}^{\otimes s_1} \otimes g^{\otimes s_2})\Phi(K \cap \bar{g}^{-1}L)\, \mu(\mathrm{d}\bar{g}),$$

for $K, L \in \mathscr{K}(V)$, where $\bar{G}$ is the group generated by G and the translation group, endowed with the product measure μ of ν and a translation invariant Haar measure on V, and where g is the rotational part of $\bar{g}$. (Again this notation is consistent with the special case where $\bar{G} = G_n$ is the motion group, $G = \mathrm{O}(n)$ and μ is the motion invariant Haar measure with its usual normalization as a 'product measure'.) Choosing bases and arguing as above, we find

$$k^G_{s_1,s_2}(\Phi)(K,L) = \sum_{i,j} d^{\Phi}_{ij}\, \Phi_i(K) \otimes \Psi_j(L) \tag{23}$$

for $K, L \in \mathscr{K}(V)$. Of course, the constants d^{Φ}_{ij} depend on the chosen bases and on Φ, but the operator, called *intersectional kinematic operator*,

$$\begin{aligned} k^G_{s_1,s_2} : \mathrm{TVal}^{s_1+s_2,G}(V) &\to \mathrm{TVal}^{s_1,G}(V) \otimes \mathrm{TVal}^{s_2,G}(V) \\ \Phi &\mapsto \sum_{i,j} d^{\Phi}_{ij}\, \Phi_i \otimes \Psi_j, \end{aligned}$$

is independent of these choices.

In the following, we explain the connection between these operators and then provide explicit examples.

Let V be a Euclidean vector space with scalar product $\langle \cdot\,,\cdot \rangle$. For $s \le r$ we define the contraction map by

$$\begin{aligned} \mathrm{contr} : V^{\otimes s} \times V^{\otimes r} &\to V^{\otimes (r-s)}, \\ (v_1 \otimes \ldots \otimes v_s, w_1 \otimes \ldots \otimes w_r) &\mapsto \langle v_1, w_1 \rangle \langle v_2, w_2 \rangle \ldots \langle v_s, w_s \rangle w_{s+1} \otimes \cdots \otimes w_r, \end{aligned}$$

and linearity. This map restricts to a map $\mathrm{contr} : \mathrm{Sym}^s V \times \mathrm{Sym}^r V \to \mathrm{Sym}^{r-s} V$. In particular, if $r = s$, the map $\mathrm{Sym}^s V \times \mathrm{Sym}^s V \to \mathbb{R}$ is the usual scalar product on $\mathrm{Sym}^s V$, which will also be denoted by $\langle \cdot\,,\cdot \rangle$.

The trace map $\mathrm{tr} : \mathrm{Sym}^s V \to \mathrm{Sym}^{s-2} V$ is defined by restriction of the map $V^{\otimes s} \to V^{\otimes (s-2)}$, $v_1 \otimes \ldots \otimes v_s \mapsto \langle v_1, v_2 \rangle v_3 \otimes \ldots \otimes v_s$, for $s \ge 2$.

The scalar product on $\mathrm{Sym}^s V$ induces an isomorphism $q^s : \mathrm{Sym}^s V \to (\mathrm{Sym}^s V)^*$ and we set

$$\begin{aligned} \mathrm{pd}^s_c : \mathrm{TVal}^{s,\infty} = \mathrm{Val}^\infty \otimes \mathrm{Sym}^s V &\xrightarrow{\mathrm{pd}_c \otimes q^s} (\mathrm{Val}^\infty)^* \otimes (\mathrm{Sym}^s V)^* = (\mathrm{TVal}^{s,\infty})^*, \\ \mathrm{pd}^s_m : \mathrm{TVal}^{s,\infty} = \mathrm{Val}^\infty \otimes \mathrm{Sym}^s V &\xrightarrow{\mathrm{pd}_m \otimes q^s} (\mathrm{Val}^\infty)^* \otimes (\mathrm{Sym}^s V)^* = (\mathrm{TVal}^{s,\infty})^*. \end{aligned}$$

From Proposition 2.7 it follows easily that

$$\mathrm{pd}^s_m = (-1)^s\, \mathrm{pd}^s_c\,. \tag{24}$$

Finally, we write

$$m, c : \mathrm{TVal}^{s_1,\infty}(V) \otimes \mathrm{TVal}^{s_2,\infty}(V) \to \mathrm{TVal}^{s_1+s_2,\infty}(V)$$

for the maps corresponding to the product and the convolution.

Theorem 2.10. *Let G be a compact subgroup of* $\mathrm{O}(n)$ *acting transitively on the unit sphere. Then the diagram*

$$\begin{array}{ccc}
\mathrm{TVal}^{s_1+s_2,G} & \xrightarrow{a^G_{s_1,s_2}} & \mathrm{TVal}^{s_1,G} \otimes \mathrm{TVal}^{s_2,G} \\
\Big\downarrow \mathrm{pd}_c^{s_1+s_2} & & \Big\downarrow \mathrm{pd}_c^{s_1} \otimes \mathrm{pd}_c^{s_2} \\
\left(\mathrm{TVal}^{s_1+s_2,G}\right)^* & \xrightarrow{c^*_G} & \left(\mathrm{TVal}^{s_1,G}\right)^* \otimes \left(\mathrm{TVal}^{s_2,G}\right)^* \\
\Big\downarrow \mathbb{F}^* & & \Big\downarrow \mathbb{F}^* \otimes \mathbb{F}^* \\
\left(\mathrm{TVal}^{s_1+s_2,G}\right)^* & \xrightarrow{m^*_G} & \left(\mathrm{TVal}^{s_1,G}\right)^* \otimes \left(\mathrm{TVal}^{s_2,G}\right)^* \\
\Big\uparrow \mathrm{pd}_m^{s_1+s_2} & & \Big\uparrow \mathrm{pd}_m^{s_1} \otimes \mathrm{pd}_m^{s_2} \\
\mathrm{TVal}^{s_1+s_2,G} & \xrightarrow{k^G_{s_1,s_2}} & \mathrm{TVal}^{s_1,G} \otimes \mathrm{TVal}^{s_2,G}
\end{array}$$

commutes and encodes the relations between product, convolution, Alesker-Fourier transform, intersectional and additive kinematic formulas.

This diagram allows us to express the additive kinematic operator in terms of the intersectional kinematic operator, and vice versa, with the Fourier transform as the link between these operators.

Corollary 2.11. *Intersectional and additive kinematic formulas are related by the Alesker-Fourier transform in the following way:*

$$a^G = \left(\mathbb{F}^{-1} \otimes \mathbb{F}^{-1}\right) \circ k^G \circ \mathbb{F},$$

or equivalently

$$k^G = (\mathbb{F} \otimes \mathbb{F}) \circ a^G \circ \mathbb{F}^{-1}.$$

This follows by looking at the outer square in Theorem 2.11, by carefully taking into account the signs coming from (24).

3.2 Some Explicit Examples of Kinematic Formulas

We start with a description of a Crofton formula for tensor valuations. Combining the connection between Crofton formulae and the product of valuations (see [4, (2) and (16)]) and the explicit formulas for the product of tensor valuations, we obtain

$$\int_{A(n,n-l)} \Phi_k^s(K\cap E)\,\mu_{n-l}(\mathrm{d}E) = \begin{bmatrix} n \\ l \end{bmatrix}^{-1} \left(\Phi_k^s \cdot \Phi_l^0\right)(K)$$

$$= \begin{bmatrix} n \\ l \end{bmatrix}^{-1} \binom{k+l}{k} \frac{kl}{k+l} \sum_{a=0,2a\neq s-1}^{\lfloor \frac{s}{2} \rfloor} \frac{1}{(4\pi)^a a!}$$

$$\times \sum_{m=0}^{a} (-1)^{a-m} \binom{a}{m} \frac{\omega_{s-2m+k+l}}{\omega_{s-2m+k}\,\omega_l} Q^a \Phi_{k+l}^{s-2a}.$$

After simplification of the inner sum by means of Zeilberger's algorithm, we obtain the Crofton formula in the Φ-basis.

Theorem 2.12. *If $k,l \geq 0$ with $k+l \leq n$ and $s \in \mathbb{N}_0$, then*

$$\int_{A(n,n-l)} \Phi_k^s(K\cap E)\,\mu_{n-l}(\mathrm{d}E) = \begin{bmatrix} n \\ l \end{bmatrix}^{-1} \binom{k+l}{k} \frac{kl}{2(k+l)} \frac{1}{\Gamma\left(\frac{k+l+s}{2}\right)}$$

$$\times \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \frac{\Gamma\left(\frac{l}{2}+j\right)\Gamma\left(\frac{k+s}{2}-j\right)}{(4\pi)^j j!} Q^j \Phi_{k+l}^{s-2j}(K).$$

Comparing the trace-free part of this formula (or by inversion), we deduce the Crofton formula for the Ψ-basis, in which the result has a particularly convenient form.

Corollary 2.13. *If $k,l \geq 0$ and $k+l \leq n$, then*

$$\int_{A(n,n-l)} \Psi_k^s(K\cap E)\,\mu_{n-l}(\mathrm{d}E) = \frac{\omega_{s+k+l}}{\omega_{s+k}\,\omega_l} \binom{k+l}{k} \frac{kl}{k+l} \begin{bmatrix} n \\ l \end{bmatrix}^{-1} \Psi_{k+l}^s(K).$$

Alternatively, as observed in [9], Corollary 2.14 can be deduced from (10), and then Theorem 2.13 can be obtained as a consequence.

Thus, having now a convenient Crofton formula for tensor valuations, we deduce from Hadwiger's integral geometric theorem an intersectional kinematic formula in the Ψ-basis.

Theorem 2.14. *Let $K,M \in \mathscr{K}^n$ and $j \in \{0,\dots,n\}$. Then*

$$\int_{\mathbf{G}_n} \Psi_j^s(K\cap gM)\,\mu(\mathrm{d}g) = \sum_{k=j}^{n} \frac{\omega_{s+k}}{\omega_{s+j}\,\kappa_{k-j}} \binom{k-1}{j-1} \begin{bmatrix} n \\ k-j \end{bmatrix}^{-1} \Psi_k^s(K)\,V_{n-k+j}(M).$$

Let us now prove some more refined intersectional kinematic formulas. In principle, we could also use Corollary 2.12 to find the intersectional kinematic formulas once we know the additive formulas. The problem is that (13) only gives us the value of a_{s_1,s_2} on the basis element $\phi_j^{s_1+s_2}$, but not on multiples of such basis elements with powers of the metric tensors. However, such terms appear naturally in the Fourier transform.

We therefore use Theorem 2.11, more precisely the lower square in the diagram.

In (18) we have computed the product of two tensor valuations. For fixed ranks s_1, s_2, the formula simplifies and can be evaluated in a closed form. For instance, if $1 \leq k, l$ with $k+l \leq n$, then

$$\begin{aligned} \Phi_k^3 \cdot \Phi_l^3 = & \frac{(k+1)(l+1)\Gamma\left(\frac{k+l+1}{2}\right)}{\pi^{\frac{5}{2}}(k+l+4)(k+l+2)(k+l)\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{l}{2}\right)} \cdot \\ & \cdot \left(-32\Phi_{k+l}^6\pi^3 + 8Q\Phi_{k+l}^4\pi^2 - Q^2\Phi_{k+l}^2\pi + \frac{1}{12}Q^3\Phi_{k+l}^0\right). \end{aligned} \tag{25}$$

Let us next work out the vertical arrows in the diagram of Theorem 2.11, i.e. the Poincaré duality pd_m. Again, this is a computation involving differential forms. The result is

$$\langle \mathrm{pd}_m^s(\Phi_k^s), \Phi_{n-k}^s \rangle = (-1)^s \frac{1-s}{\pi^s s!^2} \binom{n}{k} \frac{k(n-k)}{4} \frac{\Gamma\left(\frac{k+s}{2}\right)\Gamma\left(\frac{n-k+s}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}. \tag{26}$$

We now explain how to compute the intersectional kinematic formula $k_{3,3}^{O(n)}$ with this knowledge.

It is clear that there is a formula of the form

$$k_{3,3}^{O(n)}(\Phi_i^6) = \sum_{k+l=n+i} a_{n,i,k} \Phi_k^3 \otimes \Phi_l^3$$

with some constants $a_{n,i,k}$ which remain to be determined. Fix k, l with $k+l = n+i$. Using (26), we find

$$\begin{aligned} \langle \mathrm{pd}_m^3\, \Phi_k^3, \Phi_{n-k}^3 \rangle &= \frac{1}{72\pi^3} \binom{n}{k} k(n-k) \frac{\Gamma\left(\frac{k+3}{2}\right)\Gamma\left(\frac{n-k+3}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}, \\ \langle \mathrm{pd}_m^3\, \Phi_l^3, \Phi_{n-l}^3 \rangle &= \frac{1}{72\pi^3} \binom{n}{l} l(n-l) \frac{\Gamma\left(\frac{l+3}{2}\right)\Gamma\left(\frac{n-l+3}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \end{aligned}$$

and therefore

$$\begin{aligned} &\langle (\mathrm{pd}_m^3 \otimes \mathrm{pd}_m^3) \circ k_{3,3}^{O(n)}(\Phi_i^6), \Phi_{n-k}^3 \otimes \Phi_{n-l}^3 \rangle \\ &= a_{n,i,k} \frac{1}{72\pi^3} \binom{n}{k} k(n-k) \frac{\Gamma\left(\frac{k+3}{2}\right)\Gamma\left(\frac{n-k+3}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \frac{1}{72\pi^3} \binom{n}{l} l(n-l) \frac{\Gamma\left(\frac{l+3}{2}\right)\Gamma\left(\frac{n-l+3}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}. \end{aligned}$$

On the other hand, by (25) and (26),

$$
\begin{aligned}
&\langle m^* \circ \mathrm{pd}_m^6(\Phi_i^6), \Phi_{n-k}^3 \otimes \Phi_{n-l}^3 \rangle = \langle \mathrm{pd}_m^6(\Phi_i^6), \Phi_{n-k}^3 \cdot \Phi_{n-l}^3 \rangle \\
&= \frac{(n-k+1)(n-l+1)\Gamma\left(\frac{n-i+1}{2}\right)}{\pi^{\frac{5}{2}}(n-i+4)(n-i+2)(n-i)\Gamma\left(\frac{n-l}{2}\right)\Gamma\left(\frac{n-k}{2}\right)} \cdot \\
&\quad \cdot \left\langle \mathrm{pd}_m^6(\Phi_i^6), -32\Phi_{n-i}^6\pi^3 + 8Q\Phi_{n-i}^4\pi^2 - Q^2\Phi_{n-i}^2\pi + \frac{1}{12}Q^3\Phi_{n-i}^0 \right\rangle \\
&= \frac{1}{207360} \frac{(k-n-1)(i-k-1)\Gamma\left(\frac{n+1}{2}\right)(i+1)(i-1)(i-3)}{\pi^5\Gamma\left(\frac{i+1}{2}\right)\Gamma\left(\frac{n-k}{2}\right)\Gamma\left(\frac{k-i}{2}\right)}.
\end{aligned}
$$

From this, the explicit value of $a_{n,i,k}$ given in the theorem follows. Comparing these expressions, we find that

$$
a_{n,i,k} = \frac{(i+1)(i-1)(i-3)}{40\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{i+1}{2}\right)} \frac{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{l}{2}\right)}{(k+1)(l+1)}
$$

The same technique can be applied to all bidegrees, but it seems hard to find a closed formula which is valid simultaneously in all cases.

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