Geometry in the non-archimedean world

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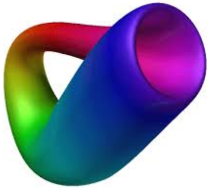
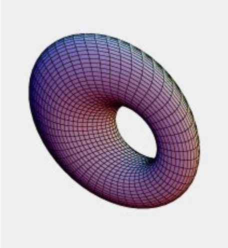
The fields $\mathbb{R}$ and $\mathbb{C}$ together with their absolute values are ubiquitous in mathematics.





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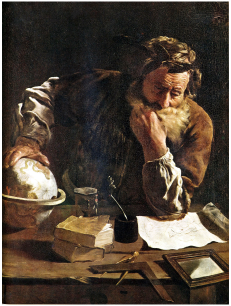
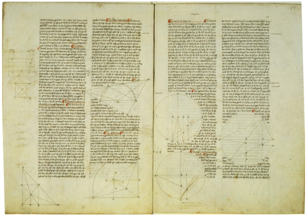
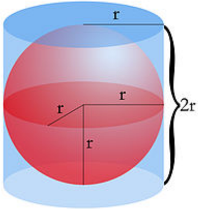


Archimedes of Syracuse (287 - 212 b.c.) as seen by Domenico Fetti (1620).

The Archimedean Axiom appears in the treatise *On the Sphere and Cylinder*





where it is shown that the volume (surface) of a sphere is two thirds of the volume (surface) of a circumscribed cylinder.
The terminology "Archimedean Axiom" was introduced in the 19th century.

Archimedean absolute values

The usual absolute values on the real and complex numbers satisfy the Archimedean axiom, i.e.

For all x, y in $\mathbb{R}$ or in $\mathbb{C}$ with $x \neq 0$ there exists a natural number n such that $|nx| > |y|$.

$p-$adic absolute value

The field $\mathbb{Q}$ of rational numbers does not only carry the real absolute value but also for every prime number p the absolute value

$$\left|\frac{n}{m}\right|_p = p^{-v_p(n)+v_p(m)},$$

where $v_p(n) =$ exponent of p in the prime factorization of n.

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$|\ n\ |_p \leqq 1$ for all natural numbers n, so that $|nx|_p \leq |x|_p$. Hence the $p-$adic absolute value violates the Archimedean axiom. We say that it is a *non-Archimedean absolute value*.

From the point of view of number theory, the real and the $p-$adic absolute values on $\mathbb{Q}$ are equally important.

- Product formula: $\prod_p \mid a \mid_p \cdot \mid a \mid_{\mathbb{R}} = 1$ for all $a \in \mathbb{Q}$.

Rational numbers

From the point of view of number theory, the real and the $p-$adic absolute values on $\mathbb{Q}$ are equally important.

- Product formula: $\prod_p \mid a \mid_p \cdot \mid a \mid_{\mathbb{R}} = 1$ for all $a \in \mathbb{Q}$.
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- $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to $\mid \ \mid_{\mathbb{R}}$. Let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ with respect to $\mid \ \mid_p$.
  Then we sometimes have a Local-Global-Principle, e.g. in the theorem of Hasse-Minkowski:

The quadratic equation $a_1 X_1^2 + a_2 X_2^2 + \ldots + a_n X_n^2 = 0$ with $a_i \in \mathbb{Q}$ has a nontrivial solution in $\mathbb{Q}^n$ if and only if it has a non-trivial solution in $\mathbb{R}^n$ and a non-trivial solution in all $\mathbb{Q}_p^n$.

Convergence

Let's do calculus in $\mathbb{Q}_p$. We define convergence of sequences and infinite sums in $\mathbb{Q}_p$ as the real case.

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Then a popular error becomes true:
$\sum_{n=1}^{\infty} a_n$ converges for the $p-$adic absolute value if and only if $\mid a_n \mid_p \to 0$.

Triangles

The $p-$adic absolute value satisfies the strong triangle inequality.

$$\mid a+b\mid_p \leqq \max\{\mid a\mid_p, \mid b\mid_p\}.$$

This follows from $v_p(m+n) \geqq \min\{v_p(m), v_p(n)\}$.

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$$| a + b |_p \leqq \max\{| a |_p, | b |_p\}.$$

This follows from $v_p(m + n) \geqq \min\{v_p(m), v_p(n)\}$. Moreover, if $| a |_p \neq | b |_p$, we find

$$| a + b |_p = \max\{| a |_p, | b |_p\}.$$

Hence all $p-$adic triangles are isosceles, i.e. at least two sides have equal length.

Balls

$p-$adic balls: $\quad a \in \mathbb{Q}_p, r > 0.$

$D^0(a,r) = \{x \in \mathbb{Q}_p : \mid x - a \mid_p < r\}$ "open ball"

$D(a,r) = \{x \in \mathbb{Q}_p : \mid x - a \mid_p \leqq r\}$ "closed ball"

$K(a,r) = \{x \in \mathbb{Q}_p : \mid x - a \mid_p = r\}$ circle.

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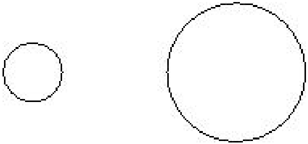
Why? If $\mid x - b \mid_p \leqq r$, then
$\mid x - a \mid_p \leqq \max\{\mid x - b \mid_p, \mid b - a \mid_p\} \leqq r$. Hence every point in a $p-$ adic ball is a center.

Balls

Therefore two $p-$adic balls are either disjoint



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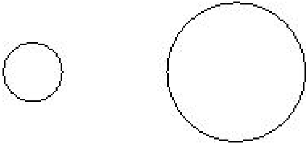
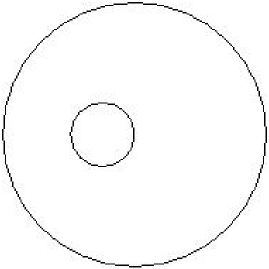


or nested



Similary, for every $b \in K(a,r)$, i.e. $\mid b-a\mid_p = r$ we find $D^0(b,r) \subset K(a,r)$.

Hence the circle is open and all closed balls are open in the $p-$adic topology.

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Bad topological news: $\mathbb{Q}_p$ is totally disconnected, i.e. the connected components are the one-point-sets.

How can we do analysis? Defining analytic functions by local expansion in power series leads to indesirable examples:

$$f(x) = \begin{cases} 1 & \text{on } D^0(0,1) \\ 0 & \text{on } K(0,1) \end{cases} .$$

In the 1960's John Tate defined rigid analytic spaces by only admitting "admissible" open coverings.

Since 1990 Vladimir Berkovich develops his approach to $p-$adic analytic spaces.

Advantage: Berkovich analytic spaces have nice topological properties.

Trick: Fill the holes in the totally disconnected $p-$adic topology with new points.

Non-archimedean fields

Let K be any field endowed with an absolute value $|\ |: K \to \mathbb{R}_{>0}$ satisfying

i) $|\ a\ | = 0$ if and only $a = 0$

ii) $|\ ab\ | = |\ a\ | \cdot |\ b\ |$

iii) $|\ a + b\ | \leqq \max\{|\ a\ |, |\ b\ |\}$.

Then $|\ |$ is a non-archimedean absolute value.

We assume that K is complete, i.e. that every Cauchy sequence in K has a limit. Otherwise replace K by its completion.

Examples:

- $\mathbb{Q}_p$ for any prime number p
- finite extensions of $\mathbb{Q}_p$
- $\mathbb{C}_p =$ completion of the algebraic closure of $\mathbb{Q}_p$.
- k any field, $0 < r < 1$.
  $k((X)) = \{\sum_{i \geqq i_0} a_i X^i : a_i \in k, i_0 \in \mathbb{Z}\}$ field of formal Laurent series with $|\sum_{i \geqq i_0} a_i X^i| = r^{i_0}$, if $a_{i_0} \neq 0$.
- k any field, $|x| = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$ trivial absolute value.

Berkovich unit disc

As above, we put $D(a, r) = \{x \in K : \mid x - a \mid \leq r\}$ for $a \in K, r > 0$.

We want to define Berkovich's unit disc.

Tate algebra

$$T = \{ \sum_{n=0}^{\infty} c_n z^n : \sum_{n=0}^{\infty} c_n a^n \text{ converges for every } a \in D(0,1)\}.$$

For every element in T we have $\mid c_n \mid \longrightarrow 0$.

Gauss norm

$$\| \sum_{n=0}^{\infty} c_n z^n \| = \max_{n \geqq 0} \mid c_n \mid .$$

Properties:

i) The Gauss norm on T is multiplicative: $\| f\ g \| = \| f \| \| g \|$

ii) It satisfies the strong triangle inequality
$\| f + g \| \leqq \max\{\| f \|, \| g \|\}$.

iii) T is complete with respect to $\| \quad \|$, hence a non-archimedean Banach algebra.

iv) Let $\overline{K}$ be the algebraic closure of K. Then
$\| f \| = \sup_{a \in \overline{K}, |a| \leqq 1} | f(a) |$

Berkovich disc

Definition

The Berkovich spectrum $\mathcal{M}(T)$ is defined as the set of all non-trivial multiplicative seminorms on T bounded by the Gauss norm.

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Hence $\mathcal{M}(T)$ consists of all maps $\gamma : T \longrightarrow \mathbb{R}_{\geqq 0}$ such that

i) $\gamma \neq 0$

ii) $\gamma(fg) = \gamma(f)\gamma(g)$

iii) $\gamma(f+g) \leqq \max\{\gamma(f), \gamma(g)\}$

iv) $\gamma(f) \leqq \| f \|$ for all $f \in T$.

It follows that $\gamma(a) = | a |$ for all $a \in K$.

Berkovich disc

For all $a \in D(0,1)$ the map

$$\begin{array}{rcl} \zeta_a : T & \longrightarrow & \mathbb{R}_{\geq 0} \\ f & \longmapsto & |\, f(a)\, | \end{array}$$

is in $\mathcal{M}(T)$.
The map $D(0,1) \to \mathcal{M}(T), a \mapsto \zeta_a$ is injective. Hence we regard the unit disc in K as a part of $\mathcal{M}(T)$. Every such point is called a point of type 1.

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$\mathcal{M}(T)$ carries a natural topology, namely the weakest topology such that all evaluation maps

$$\begin{array}{rcl} \mathcal{M}(T) & \longrightarrow & \mathbb{R} \\ \gamma & \longmapsto & \gamma(f) \end{array}$$

for $f \in T$ are continuous.

Berkovich disc

The restriction of this topology to $D(0,1)$ is the one given by the absolute value on K, hence it is disconnected on $D(0,1)$.

The whole topological space $\mathcal{M}(T)$ however has nice nonnectedness properties. It contains additional points "filling up the holes" in $D(0,1)$.

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Assume that $|\ |$ is not the trivial absolute value and (for simplicity) that K is algebraically closed.

Berkovich disc

Lemma

Let $a \in D(0,1)$ and r a real number with $0 < r \leqq 1$.
Then the supremum norm over $D(a,r)$

$$\begin{array}{rcl} \zeta_{a,r} : T & \longrightarrow & \mathbb{R}_{\geqq 0} \\ f & \longmapsto & \sup\limits_{x \in D(a,r)} \mid f(x) \mid \end{array}$$

is a point in $\mathcal{M}(T)$.

Example: The Gauss norm $\zeta_{0,1}$.

Berkovich disc

Hence the seminorms ζ_a for $a \in D(0,1)$ and the norms $\zeta_{a,r}$ for $a \in D(0,1)$ lie in $\mathcal{M}(T)$.
For some fields, we have to add limits of $\zeta_{a,r}$ along a decreasing sequence of nested discs in order to get all points in $\mathcal{M}(T)$.

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Theorem

$\mathcal{M}(T)$ is a compact Hausdorff space and uniquely path-connected.

Paths in the Berkovich disc

Take $a \in D(0,1)$ and let ζ_a be the associated point of type 1. We put $\zeta_a = \zeta_{a,0}$. Then the map

$$\begin{array}{rcl} [0,1] & \longrightarrow & \mathcal{M}(T) \\ r & \longmapsto & \zeta_{a,r} \end{array}$$

is continuous. Its image is a path $[\zeta_a, \zeta_{0,1}]$ from ζ_a to $\zeta_{a,1} = \zeta_{0,1}$ $\big($since $D(a,1) = D(0,1)\big)$.

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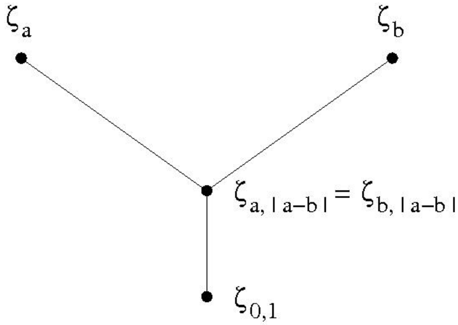
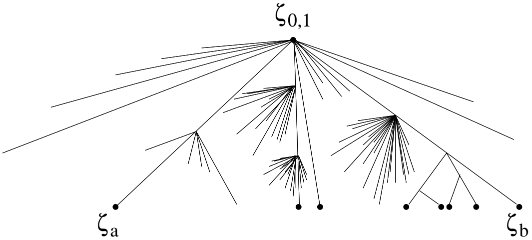
Let $b \in D(0,1)$ be a second point. Then $\zeta_{a,r} = \zeta_{b,r}$ if and only if $D(a,r) = D(b,r)$, hence if and only if $\mid a - b \mid \leqq r$.
Hence the paths $[\zeta_a, \zeta_{0,1}]$ and $[\zeta_b, \zeta_{0,1}]$ meet in $\zeta_{a,|a-b|} = \zeta_{b,|a-b|}$ and travel together to the Gauss point from there on.

Paths in the Berkovich disc



Berkovich disc

We can visualize $\mathcal{M}(T)$ as a tree which has infinitely many branches growing out of every point contained in a dense subset of any line segment. Branching occurs only at the points $\zeta_{a,r}$ for $r \in |K^\times|$.



General theory: Put $z = (z_1, \ldots, z_n)$ and define the Tate algebra as

$$T_n = \{\sum_I a_I z^I :\mid a_I \mid \underset{|I|\to\infty}{\longrightarrow} 0\}.$$

A quotient $\varphi : T_n \twoheadrightarrow A$ together with the residue norm

$$\| f \|_A = \inf_{\varphi(g)=f} \| g \|$$

is called a (strict) K-affinoid algebra.

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The Berkovich spectrum $\mathcal{M}(A)$ is the set of bounded multiplicative seminorms on A.

Berkovich spaces

An analytic space is a topological space with a covering by $\mathcal{M}(A)'s$ together with a suitable sheaf of analytic functions.

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Every scheme Z of finite type over K (i.e. every set of solutions of a number of polynomial equations in several variables over K) induces a Berkovich analytic space Z^{an}.

Theorem

i) Z is connected if and only if Z^{an} is pathconnected.
ii) Z is separated if and only if Z^{an} is Hausdorff.
iii) Z is proper if and only if Z^{an} is (Hausdorff and) compact.

Berkovich spaces

Berkovich spaces have found a variety of applications, e.g. (we apologize for any contributions which we have overlooked)

- to prove a conjecture of Deligne on vanishing cycles (Berkovich)
- in local Langlands theory (Harris-Taylor)
- to develop a $p-$adic avatar of Grothendieck's "dessins d'enfants" (André)
- to develop a $p-$adic integration theory over genuine paths (Berkovich)
- in potential theory and Arakelov Theory (Baker/Rumely, Burgos/Philippon/Sombra, Chambert-Loir, Favre/Jonsson, Thuillier,...)

and also

- in inverse Galois theory (Poineau)
- in the study of Bruhat-Tits buildings (Rémy/Thuillier/W.)
- in the new field of tropical geometry (Baker, Gubler, Payne, Rabinoff, W., . . .)
- in settling some cases of the Bogomolov conjecture (Gubler, Yamaki)
- in Mirror Symmetry via non-archimedean degenerations (Kontsevich/Soibelman, Mustata/Nicaise)
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Let's look forward to other interesting results in the future!