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ORBIFOLD POINTS ON PRYM-TEICHMÜLLER CURVES IN GENUS THREE

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Abstract. Prym-Teichmüller curves $W_D(4)$ constitute the main examples of primitive Teichmüller curves in the moduli space $\mathcal{M}_3$. We determine, for each non-square discriminant $D > 1$, the number and type of orbifold points in $W_D(4)$. These results, together with the formulas of Lanneau-Nguyen and Möller for the number of cusps and the Euler characteristic, complete the topological characterisation of Prym-Teichmüller curves in genus 3.

Crucial for the determination of the orbifold points is the analysis of families of genus 3 cyclic covers of degree 4 and 6, branched over four points of $\mathbb{P}^1$. As a side product of our study, we provide an explicit description of the Jacobians and the Prym-Torelli images of these two families.

Contents

1. Introduction

A *Teichmüller curve* is an algebraic curve in the moduli space $\mathcal{M}_g$ of genus g curves that is totally geodesic for the Teichmüller metric. Teichmüller curves arise naturally from *flat surfaces*, i.e. elements (X, ω) of the bundle $\Omega\mathcal{M}_g$ over $\mathcal{M}_g$, consisting of a curve X with a holomorphic 1-form $\omega \in \Omega(X)$. The bundle $\Omega\mathcal{M}_g$ is endowed with an $\mathrm{SL}_2(\mathbb{R})$-action, defined by affine shearing of the flat structure

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induced by the differential. In the rare case that the closure of the projection to $\mathcal{M}_g$ of the $\mathrm{SL}_2(\mathbb{R})$-orbit of an element (X, ω) is an algebraic curve, i.e. that (X, ω) has many real symmetries, we obtain a Teichmüller curve.

Only few examples of families of (primitive) Teichmüller curves are known, see [McM07], [McM06], [KS00] and [BM10]. In genus 2, McMullen was able to construct the *Weierstraß curves*, and thereby classify all Teichmüller curves in $\mathcal{M}_2$ by analysing when the Jacobian of the flat surface admits real multiplication. However, for larger genus requiring real multiplication on the entire Jacobian is too strong a restriction. By relaxing this condition he constructed the *Prym-Teichmüller curves* $W_D(4)$ in genus 3 and $W_D(6)$ in genus 4 (see Section 2 for definitions).

While the situation for genus 2 is fairly well understood, things are less clear for higher genus. As curves in $\mathcal{M}_g$, Teichmüller curves carry a natural orbifold structure. As such, one is primarily interested in their homeomorphism type, i.e. the genus, the number of cusps and the number and type of orbifold points. In genus two, this was solved for the Weierstraß curves by McMullen [McM05], Bainbridge [Bai07] and Mukamel [Muk14].

For the Prym-Teichmüller curves in genus 3 and 4 the Euler characteristics were calculated by Möller [Möl14] and the number of cusps were counted by Lanneau and Nguyen [LN14]. The primary aim of this paper is to describe the number and type of orbifold points occurring in genus 3, thus completing the topological characterisation of $W_D(4)$ for all (non-square) discriminants D via the formula

$$2 - 2g = \chi + C + \sum_d h_d \left(1 - \frac{1}{d}\right) \tag{1}$$

where g denotes the genus of $W_D(4)$, χ the Euler characteristic, C the number of cusps and h_d the number of orbifold points of order d.

Theorem 1.1. *The Prym-Teichmüller curves $W_D(4)$ for genus three have orbifold points of order 2, 3, 4 or 6. For non-square discriminant D, the number of such points on $W_D(4)$ is given by the formulas $h_2(D)$, $h_3(D)$, $h_4(D)$ and $h_6(D)$ defined in Section 5.*

This will be the content of Theorem 5.1 and Theorem 5.6. The topological invariants of $W_D(4)$ for D up to 300 are given in Table 2 on page 35.

Our approach to solving this problem is purely algebraic and therefore the use of tools from the theory of flat surfaces will be sporadic.

Two families of curves will play a special role in determining orbifold points on Prym-Teichmüller curves, namely the *Wollmilchsau family* and the *C_6-family*, which we will introduce in Section 3. They parametrise certain genus 3 cyclic covers of $\mathbb{P}^1$ of degree 4 and 6, respectively. There are two special points in these families, namely the *Fermat curve* of degree 4, which is the only element of the Wollmilchsau family with a cyclic group of automorphisms of order 8, and the exceptional *Wiman curve* of genus 3, which is the unique intersection of the two families and the unique curve in genus 3 that admits a cyclic group of automorphisms of order 12.

The fact that orbifold points in $W_D(4)$ correspond to points of intersection with these two families will follow from the study of the action of the Veech group $\mathrm{SL}(X, \omega)$ carried out in Section 2. A consequence of this study is that orbifold points of order 4 and 6 correspond to the Fermat and Wiman curves respectively, while points of order 2 and 3 correspond to generic intersections with the Wollmilchsau family and the C_6-family, respectively.

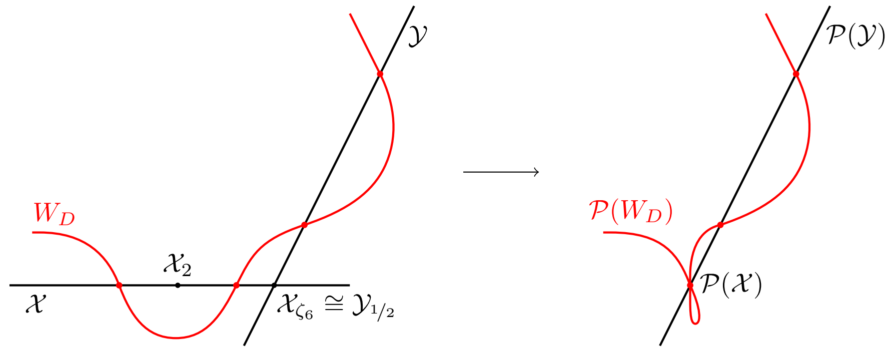

FIGURE 1. The Wollmilchsau family, the C_6-family, and the curve W_D inside $\mathcal{M}_3$ and their image under the Prym-Torelli map in $\mathcal{A}_{2,(1,2)}$.

In order to determine these points of intersection, we will need a very precise description of the two families or, more precisely, of their images under the Prym-Torelli map. To this end, we explicitly compute the period matrices of the two families in Section 4. While the analysis of different types of orbifold points was rather uniform up to this point, the Wollmilchsau family and the C_6-family behave quite differently under the Prym-Torelli map. In particular, the Prym-Torelli image of the Wollmilchsau family is constant.

Theorem 1.2. *The Prym-Torelli image of the Wollmilchsau family $\mathcal{X}$ is isogenous to the point $E_{\mathrm{i}} \times E_{\mathrm{i}}$ in the moduli space $\mathcal{A}_{2,(1,2)}$ of abelian surfaces with $(1,2)$-polarisation, where E_{i} denotes the elliptic curve with complex multiplication by $\mathbb{Q}[\mathrm{i}]$. Orbifold points of $W_D(4)$ of order 2 and 4 correspond to intersections with this family.*

In contrast, the image of the C_6-family under the Prym-Torelli map lies in the Shimura curve of discriminant 6. We show this by giving a precise description of the endomorphism ring of the general member of this family.

Theorem 1.3. *The closure of the Prym-Torelli image of the C_6-family $\mathcal{Y}$ in $\mathcal{A}_{2,(1,2)}$ is the (compact) Shimura curve $\mathbb{H}/\Delta(2,6,6)$ defined in Proposition 4.6. The generic element of the family has period matrix and polarisation as in Proposition 4.5 and its endomorphism ring is isomorphic to the maximal order in the indefinite rational quaternion algebra of discriminant 6. Orbifold points of $W_D(4)$ of order 3 and 6 correspond to intersections with this family.*

The relationship between the Wollmilchsau family, the C_6-family, and a Prym-Teichmüller curve is illustrated in Figure 1.

In Section 5, we finally determine the intersections of the Prym-Teichmüller curve $W_D(4)$ with the Wollmilchsau family and the C_6-family by studying which points in their Prym-Torelli images admit real multiplication by the quadratic order $\mathcal{O}_D$ and by determining the corresponding eigenforms for this action. An immediate consequence is the following result.

Corollary 1.4. *The only Prym-Teichmüller curves in $\mathcal{M}_3$ with orbifold points of order 4 or 6 are $W_8(4)$ of genus zero with one cusp, one point of order 3 and one point of order 4, and $W_{12}(4)$ of genus zero with two cusps and one point of order 6.*

Note that our approach is similar to that of Mukamel in [Muk14], although it differs in almost every detail. In the following we give a brief summary of the techniques he used to classify orbifold points of Weierstraß curves in genus 2, to illustrate the similarities with and differences to our case.

The first difference is that, while in genus 2 all curves are hyperelliptic, this is never the case for genus 3 curves on Prym-Teichmüller curves by Lemma 2.7. Luckily, the Prym involution is a satisfactory substitute in all essential aspects. In particular, while Mukamel obtains restrictions on the types of orbifold points in genus 2 by observing the action on the Weierstraß points, we acquire an analogous result in genus 3 by relating symmetries of Prym forms to automorphisms of elliptic curves (Proposition 2.1).

From this point onward, however, the genus 2 and 3 situations begin to drift apart. Mukamel shows that the orbifold points on genus 2 Weierstraß curves correspond to curves admitting an embedding of the dihedral group D_8 into their automorphism group and whose Jacobian admits complex multiplication. He then identifies the space of genus 2 curves admitting a faithful D_8 action with the modular curve $\mathbb{H}/\Gamma_0(2)$. In this model, the curves admitting complex multiplication are well-known to correspond to the imaginary quadratic points in the fundamental domain. Thus counting orbifold points in genus 2 is equivalent to computing class numbers of imaginary quadratic fields. Moreover, this period domain permits associating concrete flat surfaces to the orbifold points via his "pinwheel" construction.

By contrast, in genus 3, each orbifold point may lie on the Wollmilchsau family *or* the C_6-family (Proposition 3.1). As mentioned above, these two cases behave quite differently. Moreover, in genus 3, we are no longer dealing with the entire Jacobian, but only with the Prym part, i.e. part of the Jacobian collapses and the remainder carries a non-principal $(1, 2)$ polarisation (see Section 2). In particular, while in Mukamel's case the appearing abelian varieties could all be obtained by taking products of elliptic curves, we are forced to construct our Jacobians "from scratch" via Bolza's method (Section 4). This adds a degree of subtlety to pinpointing the actual intersection points of the Wollmilchsau family and the C_6-family with a given $W_D(4)$. One consequence is that the class numbers determining the number of orbifold points in our case are associated to slightly more involved quadratic forms (Section 5).

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2. Orbifold points on Prym-Teichmüller curves

The aim of this section is to prove the following statement.

Proposition 2.1. *A flat surface (X, ω) parametrised by a point in $W_D(4)$ is an orbifold point of order n if and only if there exists $\sigma \in \mathrm{Aut}(X)$ of order $2n$ satisfying $\sigma^*\omega = \zeta_{2n}\omega$, where ζ_{2n} is some primitive order $2n$ root of unity.*

The different possibilities are listed in Table 1.

Before proceeding with the proof, we briefly recall some notation and background information.

	$\mathrm{ord}(\sigma)$	Branching data
(i)	4	$(0; 4, 4, 4, 4)$
(ii)	6	$(0; 2, 3, 3, 6)$
(iii)	8	$(0; 4, 8, 8)$
(iv)	12	$(0; 3, 4, 12)$

TABLE 1. Possible orders of σ and their corresponding branching data.

Orbifold Points. If G is a finite group acting on a Riemann surface X of genus $g \geq 2$, we define the *branching data* (or signature of the action) as the signature of the orbifold quotient X/G, that is $\Sigma \coloneqq (\gamma; m_1, \dots, m_r)$, where γ is the genus of the quotient X/G and the projection is branched over r points with multiplicities m_i.

Recall that an *orbifold point* of an orbifold $\mathbb{H}/\Gamma$ is the projection of a fixed point of the action of Γ, i.e. a point $s \in \mathbb{H}$ so that $\mathrm{Stab}_\Gamma(s) = \{A \in \Gamma : A \cdot s = s\}$ is strictly larger than the kernel of the action of Γ. Observe that this is equivalent to requiring the image of $\mathrm{Stab}_\Gamma(s)$ in $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{Aut}(\mathbb{H})$, which we denote by $\mathbb{P}\mathrm{Stab}_\Gamma(s)$, to be non-trivial. We call the cardinality of $\mathbb{P}\mathrm{Stab}_\Gamma(s)$ the *(orbifold) order* of s.

In the case of a Teichmüller curve, the close relationship between the uniformising group Γ and the affine structure of the fibres permits a characterisation of orbifold points in terms of flat geometry. To make this precise, we need some more notation.

Teichmüller curves. Recall that a *flat surface* (X, ω) consists of a curve X together with a holomorphic differential form ω on X, which induces a flat structure by integration. Hence we may consider the moduli space of flat surfaces $\Omega\mathcal{M}_g$ as a bundle over the moduli space of genus g curves $\mathcal{M}_g$. Recall that there is a natural $\mathrm{SL}_2(\mathbb{R})$ action on $\Omega\mathcal{M}_g$ by shearing the flat structure, which respects – in particular – the zeros of the differentials. Every Teichmüller curve arises as the projection to $\mathcal{M}_g$ of the (closed) $\mathrm{SL}_2(\mathbb{R})$ orbit of some (X, ω). As $\mathrm{SO}(2)$ acts holomorphically on the fibres, we obtain the following commutative diagram

$$\begin{array}{ccc} \mathrm{SL}_2(\mathbb{R}) & \xrightarrow{F} & \Omega\mathcal{M}_g \\ \downarrow & & \downarrow \\ \mathbb{H} \cong \mathrm{SO}(2)\backslash \mathrm{SL}_2(\mathbb{R}) & \xrightarrow{f} & \mathbb{P}\Omega\mathcal{M}_g \\ \downarrow & & \downarrow \pi \\ \mathcal{C} = \mathbb{H}/\Gamma & \longrightarrow & \mathcal{M}_g \end{array}$$

where the map F is given by the action $A \mapsto A \cdot (X, \omega)$ and $\mathcal{C}$ is uniformised by

$$\Gamma = \mathrm{Stab}(f) \coloneqq \{A \in \mathrm{SL}_2(\mathbb{R}) : f(A \cdot t) = f(t)\ ,\ \forall t \in \mathbb{H}\} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot \mathrm{SL}(X, \omega) \cdot \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

Here, $\mathrm{SL}(X, \omega)$ is the *affine group* of (X, ω), i.e. the derivatives of homeomorphisms of X that are affine with regard to the flat structure.

Given $t \in \mathbb{H}$, we will write $A_t \in \mathrm{SL}_2(\mathbb{R})$ for (a representative of) the corresponding element in $\mathrm{SO}(2)\backslash \mathrm{SL}_2(\mathbb{R})$ and (X_t, ω_t) for (a representative of) $f(t) = [A_t \cdot (X, \omega)] \in \mathbb{P}\Omega\mathcal{M}_g$.

For proofs and details, see e.g. [Möl11b], [Kuc12], [McM03].

In the following, we will be primarily interested in a special class of Teichmüller curves.

Prym-Teichmüller curves. To ensure that the $\mathrm{SL}_2(\mathbb{R})$ orbit of a flat surface is not too large, the flat structure must possess sufficient real symmetries. McMullen observed that in many cases this can be achieved by requiring the Jacobian to admit real multiplication that "stretches" the differential. However, it turns out that for genus greater than 2, requiring the whole Jacobian to admit real multiplication is too strong a restriction.

More precisely, for positive $D \equiv 0, 1 \bmod 4$ non-square, we denote by $\mathcal{O}_D = \mathbb{Z}[T]/(T^2 + bT + c)$ with $D = b^2 - 4c$, the unique (real) quadratic order associated to D and say that a (polarised) abelian surface A has *real multiplication* by $\mathcal{O}_D$ if it admits an embedding $\mathcal{O}_D \hookrightarrow \mathrm{End}(A)$ that is self-adjoint with respect to the polarisation. We call the real multiplication by $\mathcal{O}_D$ *proper*, if the embedding cannot be extended to any quadratic order containing $\mathcal{O}_D$.

Now, consider a curve X with an involution ρ. The projection $\pi\colon X \to X/\rho$ induces a morphism $\mathrm{Jac}(\pi)\colon \mathrm{Jac}(X) \to \mathrm{Jac}(X/\rho)$ of the Jacobians and we call the kernel $\mathcal{P}(X,\rho)$ of $\mathrm{Jac}(\pi)$ the *Prym variety* associated to (X,ρ). In the following, we will always require the Prym variety to be 2-dimensional, hence the construction only works for X of genus 2, 3 or 4. Observe that, when X has genus 3, the Prym variety $\mathcal{P}(X,\rho)$ is no longer principally polarised but carries a $(1,2)$-polarisation. See for instance [BL04, Chap. 12] or [Möl14] for details.

Starting with a flat surface (X,ω) where X admits an involution ρ satisfying $\rho^*\omega = -\omega$ and identifying $\mathrm{Jac}(X)$ with $\Omega(X)^\vee/\mathrm{H}_1(X,\mathbb{Z})$, the differential ω is mapped to the Prym part and hence, whenever $\mathcal{P}(X,\rho)$ has real multiplication by $\mathcal{O}_D$, we obtain an induced action of $\mathcal{O}_D$ on ω. We denote by $\mathcal{E}_D(2g-2) \subset \Omega\mathcal{M}_g$ the space of (X,ω) such that

1. X admits an involution ρ such that $\mathcal{P}(X,\rho)$ is 2-dimensional,
2. the form ω has a single zero and satisfies $\rho^*\omega = -\omega$, and
3. $\mathcal{P}(X,\rho)$ admits proper real multiplication by $\mathcal{O}_D$ with ω as an eigenform,

and by $\mathbb{P}\mathcal{E}_D(2g-2)$ the corresponding quotient by the $\mathrm{SO}(2)$ action. McMullen showed [McM03; McM06] that by defining $W_D(2g-2)$ as the projection of the locus $\mathcal{E}_D(2g-2)$ to $\mathcal{M}_g$, we obtain (possibly a union of) Teichmüller curves for every discriminant D in $\mathcal{M}_2$, $\mathcal{M}_3$ and $\mathcal{M}_4$. In the genus 2 case, the Prym involution is given by the hyperelliptic involution and the curve $W_D(2)$ is called the *Weierstraß curve*, while the curves $W_D(4)$ and $W_D(6)$ in $\mathcal{M}_3$ and $\mathcal{M}_4$, respectively, are known as *Prym-Teichmüller curves.* As we are primarily interested in the genus 3 case, we shall frequently refer to $W_D(4)$ simply by W_D.

We are now in a position to give a precise characterisation of orbifold points on Teichmüller curves in terms of flat geometry.

Proposition 2.2. *Let $\mathbb{H}/\Gamma$ be a Teichmüller curve generated by some $(X,\omega) = (X_\mathrm{i},\omega_\mathrm{i})$. Then the following are equivalent.*

- *The point $t \in \mathbb{H}$ projects to an orbifold point in $\mathbb{H}/\Gamma$.*
- *There exists an elliptic matrix $C \in \mathrm{SL}(X,\omega)$, $C \neq \pm 1$ such that $A_t C A_t^{-1} \in \mathrm{SO}(2)$.*
- *The corresponding flat surface (X_t,ω_t) admits a (holomorphic) automorphism σ satisfying $[\sigma^*\omega_t] = [\omega_t]$ and $\sigma^*\omega_t \neq \pm\omega_t$.*

Proof. By the above correspondence, $t \in \mathbb{H}$ corresponds to some $(X_t,[\omega_t]) \in \mathbb{P}\Omega\mathcal{M}_g$ and equivalently to some $A_t \in \mathrm{SO}(2)\backslash \mathrm{SL}_2(\mathbb{R})$ with $[A_t \cdot (X,\omega)] = (X_t,[\omega_t])$.

Now, $C \in \mathrm{SL}(X,\omega)$ is in the stabiliser of A_t if and only if there exists $B \in \mathrm{SO}(2)$ such that

$$A_t C = B A_t, \text{ i.e. } A_t C A_t^{-1} \in \mathrm{SO}(2).$$

But then, by definition, $C \in \mathrm{SL}(X,\omega)$ is elliptic. Moreover, $C' := A_t C A_t^{-1}$ lies in $\mathrm{SL}(A_t \cdot (X,\omega)) = \mathrm{SL}(X_t,\omega_t)$, and as $C' \in \mathrm{SO}(2)$, the associated affine map is in fact a holomorphic automorphism σ of X_t. In particular, $\sigma^*\omega_t = \zeta\omega_t \in [\omega_t]$, where ζ is the corresponding root of unity.

Finally, observe that C acts trivially on $\mathrm{SO}(2)\backslash\mathrm{SL}_2(\mathbb{R})$ if and only if for every $A \in \mathrm{SL}_2(\mathbb{R})$ there exists $B \in \mathrm{SO}(2)$ so that

$$AC = BA, \text{ i.e. } ACA^{-1} \in \mathrm{SO}(2)\ \forall A \in \mathrm{SL}_2(\mathbb{R})$$

and this is the case if and only if $C = \pm 1$. □

Corollary 2.3. *There is a one-to-one correspondence between*

- *elements in* $\mathrm{Stab}_\Gamma(t)$*,*
- *elements in* $\mathrm{SL}\left(A_t \cdot (X,\omega)\right) \cap \mathrm{SO}(2)$*, and*
- *holomorphic automorphisms σ of X_t satisfying $\sigma^*\omega_t \in [\omega_t]$.*

In the case of Weierstraß and Prym-Teichmüller curves, we can say even more.

Corollary 2.4. *Let $W_D(2g-2)$ be as above, let $(X_t,[\omega_t]) \in \mathbb{P}\mathcal{E}_D(2g-2)$ correspond to an orbifold point and let σ be a non-trivial automorphism of $(X_t,[\omega_t])$. Let $\pi : X_t \to X_t/\sigma$ denote the projection. Then π has a totally ramified point.*

Proof. As $[\sigma^*\omega] = [\omega]$ and ω has a single zero, this must be a fixed point of σ, hence a totally ramified point. □

Note that the Prym-Teichmüller curves $W_D(4)$ and $W_D(6)$ lie entirely inside the branch locus of $\mathcal{M}_3$ and $\mathcal{M}_4$ respectively, as all their points admit involutions. In particular, the Prym involution ρ_t on each (X_t,ω_t) acts as -1, i.e. $\rho_t^*\omega_t = -\omega_t$, and therefore it does not give rise to orbifold points.

Corollary 2.5. *The Prym involution is the only non-trivial generic automorphism of $W_D(2g-2)$, i.e. the index $[\mathrm{Stab}_\Gamma(s) : \mathbb{P}\mathrm{Stab}_\Gamma(s)]$ is always 2.*

Moreover, Proposition 2.2 gives a strong restriction on the type of automorphisms inducing orbifold points.

Lemma 2.6. *The point in $W_D(2g-2)$ corresponding to a flat surface $(X,[\omega])$ is an orbifold point of order n if and only if $(X,[\omega])$ admits an automorphism σ of order $2n$. Moreover, σ^n is the Prym involution.*

Proof. Let $P \in X$ be the (unique) zero of ω. By the above, the automorphisms of $(X,[\omega])$ lie in the P-stabiliser of $\mathrm{Aut}(X)$. But these are (locally) rotations around P, hence the stabiliser is cyclic and of even order, as it contains the Prym involution ρ. Conversely, any automorphism σ fixing P satisfies $[\sigma^*\omega] = [\omega]$. The remaining claims follow from Corollary 2.5. □

To determine the number of branch points in the genus 3 case, we start with the following observation (cf. [Möl14, Lemma 2.1]).

Lemma 2.7. *The curve W_D is disjoint from the hyperelliptic locus in $\mathcal{M}_3$.*

Proof. Let $(X, [\omega])$ correspond to a point on W_D, denote by ρ the Prym involution on X and assume that X is hyperelliptic with involution σ. As X is of genus 3, $\sigma \neq \rho$. But σ commutes with ρ and therefore $\tau := \sigma \circ \rho$ is another involution.

Recall that σ acts by -1 on all of $\Omega(X)$. Denote by $\Omega(X)^{\pm}$ the decomposition into ρ-eigenspaces. The -1 eigenspace of τ is therefore $\Omega(X)^{+}$ and the $+1$ eigenspace is $\Omega(X)^{-}$. In particular, any Prym form on X is τ invariant, i.e. a pullback from X/τ.

However, by checking the dimensions of the eigenspaces, we see that X/τ is of genus 2, hence $X \to X/\tau$ is unramified by Riemann-Hurwitz and we cannot obtain a form with a fourfold zero on X by pullback, i.e. $(X, \omega) \notin \mathcal{E}_D(4)$, a contradiction. □

We now have all we need to prove Proposition 2.1.

Proof of Proposition 2.1. Starting with Proposition 2.2 and Lemma 2.6, observe that σ descends to an automorphism $\overline{\sigma}$ of the elliptic curve X/ρ. Note that $\overline{\sigma}$ acts non-trivially, since $\sigma \neq \rho$, and it has at least one fixed point, hence $X/\sigma \cong \mathbb{P}^1$ and it is well-known that $\overline{\sigma}$ can only be of degree 2, 3, 4 or 6.

For the number of ramification points, since X has genus 3, by Riemann-Hurwitz

$$4 = -4n + 2n \sum_{d|2n} \left(1 - \frac{1}{d}\right) e_d \, ,$$

where e_d is the number of points over which σ ramifies with order d. A case by case analysis using Lemma 2.7 shows that the only possibilities are those listed in Table 1. □

Remark 2.8. *Automorphism groups of genus three curves were classified by Komiya and Kuribayashi in [KK79] (P. Henn studied them even earlier in his PhD dissertation [Hen76]). One can also find a complete classification of these automorphism groups together with their branching data in [Bro91, Table 5], including all the information in our Table 1.*

3. Cyclic covers

Proposition 2.1 classified orbifold points of W_D in terms of automorphisms of the complex curve. The aim of this section is to express these conditions as intersections of W_D with certain families of cyclic covers of $\mathbb{P}^1$ in $\mathcal{M}_3$.

Let $\mathcal{X} \to \mathbb{P}^* := \mathbb{P}^1 - \{0, 1, \infty\}$ be the family of projective curves with affine model

$$\mathcal{X}_t \colon y^4 = x(x-1)(x-t)$$

and $\mathcal{Y} \to \mathbb{P}^*$ the family of projective curves with affine model

$$\mathcal{Y}_t \colon y^6 = x^2(x-1)^2(x-t)^3.$$

The family $\mathcal{X}$ has been intensely studied, notably in [Guà01] and [HS08]. In fact, it is even a rare example of a curve that is both a Shimura and a Teichmüller curve (cf. [Möl11a]). Following [HS08], we will refer to it as the Wollmilchsau family.

The family $\mathcal{Y}$ is related to the Shimura curve of discriminant 6, which has been studied for instance in [Voi09] and [PS11]. We will refer to it as the C_6-family.

Proposition 3.1. *If $(X, [\omega])$ corresponds to an orbifold point on W_D then X is isomorphic to some fibre of $\mathcal{X}$ or $\mathcal{Y}$.*

Moreover, $(X,[\omega])$ is of order six if and only if X is isomorphic to $\mathcal{X}_{\zeta_6} \cong \mathcal{Y}_{1/2}$ the (unique) intersection point of $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{M}_3$; it is of order four if and only if X is isomorphic to $\mathcal{X}_{-1}$; it is of order two if it corresponds to a generic fibre of $\mathcal{X}$ and of order three if it corresponds to a generic fibre of $\mathcal{Y}$.

To state the converse, we need to pick a Prym eigenform on the appropriate fibres of $\mathcal{X}$ and $\mathcal{Y}$.

First, let us briefly review some well-known facts on the theory of cyclic coverings which will be applicable to both the Wollmilchsau family $\mathcal{X}$ and the C_6-family $\mathcal{Y}$. For more background and details, see for example [Roh09].

Consider the family $\mathcal{Z} \to \mathbb{P}^*$ of projective curves with affine model

$$\mathcal{Z}_t\colon y^d = x^{a_1}(x-1)^{a_2}(x-t)^{a_3},$$

and choose a_4 so that $\sum a_i \equiv 0 \bmod d$, with $0 < a_i < d$. Moreover, we will suppose $\gcd(a_1,a_2,a_3,a_4,d) = 1$ so that the curve is connected. Note that any (connected) family of cyclic covers, ramified over four points, may be described in this way.

For each fibre $\mathcal{Z}_t$ the map $\pi : (x,y) \mapsto x$ yields a cover $\mathcal{Z}_t \to \mathbb{P}^1$ of degree d ramified over 0, 1, t and ∞ with branching orders d/a_1, d/a_2, d/a_3 and d/a_4 respectively. Then, by Riemann-Hurwitz, the genus of $\mathcal{Z}_t$ is $d+1-(\sum_{i=1}^4 \gcd(a_i,d))/2$.

Note that the number of preimages of 0, 1, t and ∞ is $\gcd(a_1,d)$, $\gcd(a_2,d)$, $\gcd(a_3,d)$ and $\gcd(a_4,d)$ respectively. Denote for instance $\pi^{-1}(0) = \{P_j\}$, with $j = 0,\ldots,\gcd(a_1,d)-1$, and let us define $s_1 = d/a_1$. The following map

$$z \mapsto \left(z^{s_1}, \zeta_d^j z^{\frac{a_1}{\gcd(a_1,d)}} \sqrt[d]{(z^{s_1}-1)^{a_2}(z^{s_1}-t)^{a_3}}\right), \qquad |z| < \varepsilon$$

gives a parametrisation of a neighbourhood of P_j. In a similar way, one can find local parametrisations around the preimages of the rest of branching values.

The map π corresponds to the quotient $\mathcal{Z}_t/\langle\alpha^{\mathcal{Z}}\rangle$ by the action of the cyclic group of order d generated by the automorphism

$$\alpha^{\mathcal{Z}} := \alpha_t^{\mathcal{Z}}\colon (x,y) \mapsto (x, \zeta_d y)\,,$$

where $\zeta_d = \exp(2\pi \mathrm{i}/d)$. When there is no ambiguity we will simply write α for $\alpha^{\mathcal{Z}}$.

As a consequence, the cyclic groups acting on $\mathcal{X}_t$ and on $\mathcal{Y}_t$ are generated by the automorphisms

$$\alpha^{\mathcal{X}} := \alpha_t^{\mathcal{X}}\colon (x,y) \mapsto (x,\zeta_4 y), \text{ and}$$
$$\alpha^{\mathcal{Y}} := \alpha_t^{\mathcal{Y}}\colon (x,y) \mapsto (x,\zeta_6 y),$$

respectively.

By Lemma 2.6, the Prym involutions are given by

$$\rho^{\mathcal{X}} := \rho_t^{\mathcal{X}} := (\alpha^{\mathcal{X}})^3\colon (x,y) \mapsto (x,-y)\,, \text{ and}$$
$$\rho^{\mathcal{Y}} := \rho_t^{\mathcal{Y}} := (\alpha^{\mathcal{Y}})^3\colon (x,y) \mapsto (x,-y)\,.$$

We will denote by $\mathcal{P}(\mathcal{X}_t)$ and $\mathcal{P}(\mathcal{Y}_t)$ the corresponding Prym varieties.

Note that different fibres of the families $\mathcal{X}$ and $\mathcal{Y}$ can be isomorphic.

In fact, in the case of the Wollmilchsau family $\mathcal{X}$ any isomorphism $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ preserving the set $\{0,1,\infty\}$ lifts to isomorphisms $\mathcal{X}_t \cong \mathcal{X}_{\phi(t)}$ for each t. As a consequence our family is parametrised by $\mathbb{P}^*/\mathfrak{S}_3$, where we take the symmetric group $\mathfrak{S}_3$ to be generated by $z \mapsto 1-z$ and $z \mapsto 1/z$. The corresponding modular maps yield curves in $\mathcal{M}_3$ and $\mathcal{A}_3$.

As for the C_6-family $\mathcal{Y}$, for each $t \in \mathbb{P}^*$ the curves $\mathcal{Y}_t$ and $\mathcal{Y}_{1-t}$ are isomorphic via the map $(x, y) \mapsto (1-x, \zeta_{12} y)$, which induces the map $z \mapsto 1-z$ on $\mathbb{P}^1$. Since any isomorphism between fibres $\mathcal{Y}_t$ and $\mathcal{Y}_{t'}$ must descend to an isomorphism of $\mathbb{P}^1$ interchanging branching values of the same order, it is clear that no other two fibres are isomorphic, and therefore the family is actually parametrised by $\mathbb{P}^*/\sim$, where $z \sim 1-z$. In Section 4.2 we will give a more explicit description of this family in terms of its Prym-Torelli image.

The discussion above proves the following.

Lemma 3.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be the families defined above.*

(1) The map $\mathbb{P}^* \to \mathcal{M}_3$, $t \mapsto \mathcal{X}_t$ is of degree 6. It ramifies over $\mathcal{X}_{-1}$ that has 3 preimages $\{\mathcal{X}_t : t = -1, 1/2, 2\}$ and $\mathcal{X}_{\zeta_6}$ that has 2 preimages $\{\mathcal{X}_t : t = \zeta_6^{\pm 1}\}$.

The only fibres with a cyclic group of automorphisms or order larger than 4 are $\mathcal{X}_{-1}$ that admits a cyclic group of order 8 and $\mathcal{X}_{\zeta_6}$ that admits a cyclic group of order 12.

(2) The map $\mathbb{P}^* \to \mathcal{M}_3$, $t \mapsto \mathcal{Y}_t$ is of degree 2. It ramifies only over $\mathcal{Y}_{1/2}$ that has a single preimage.

The only fibre with a cyclic group of automorphisms of order larger than 6 is $\mathcal{Y}_{1/2}$ that admits a cyclic group of order 12.

Proof of Proposition 3.1. If $(X, [\omega])$ corresponds to an orbifold point on W_D, then X must belong to one of the families in Table 1.

First of all, note that curves of type (iii) admit an automorphism of order 4 with branching data $(0; 4, 4, 4, 4)$, and therefore they also belong to family (i). Similarly, those of type (iv) admit automorphisms of order 4 and 6 with branching data $(0; 4, 4, 4, 4)$ and $(0; 2, 3, 3, 6)$ respectively, and therefore they belong both to families (i) and (ii). As a consequence we can suppose that X belongs either to (i) or (ii).

Let us suppose that X is of type (i). Looking at the branching data, one can see that X is necessarily isomorphic to one of the following two curves for some $t \in \mathbb{P}^*$

$$y^4 = x(x-1)(x-t)\,,$$
$$y^4 = x^3(x-1)^3(x-t)\,.$$

However curves of the second kind are always hyperelliptic, with hyperelliptic involution given by

$$\tau\colon (x, y) \mapsto \left(\frac{tx - t}{x - t}, t(t-1)\frac{y}{(y-t)^2}\right).$$

As points of W_D cannot correspond to hyperelliptic curves by Lemma 2.7, the curve X is necessarily isomorphic to some $\mathcal{X}_t$.

If X is of type (ii), the branching data tells us that X must be isomorphic to some fibre $\mathcal{Y}_t$.

The claim about the order of the orbifold points follows from Lemma 2.6 and Lemma 3.2. ☐

Let us note here that the special fibre $\mathcal{X}_{-1}$ is isomorphic to the Fermat curve $x^4 + y^4 + z^4 = 0$ and that the unique intersection point of the Wollmilchsau family and the C_6-family, that is $\mathcal{X}_{\zeta_6} \cong \mathcal{Y}_{1/2}$, is isomorphic to the exceptional Wiman curve of genus 3 with affine equation $y^3 = x^4 + 1$.

3.1. **Differential forms.** By the considerations in Section 2, we are only interested in differential forms with a single zero in a fixed point of the Prym involution.

Lemma 3.3. *Let $t \in \mathbb{P}^*$.*

(1) The space of holomorphic 1-forms on each fibre $\mathcal{X}_t$ of the Wollmilchsau family is generated by the $(\alpha^{\mathcal{X}})^*$-eigenforms

$$\omega_1^{\mathcal{X}} = \frac{\mathrm{d}x}{y^3}\,, \qquad \omega_2^{\mathcal{X}} = \frac{x\mathrm{d}x}{y^3}\,, \qquad \omega_3^{\mathcal{X}} = \frac{\mathrm{d}x}{y^2}\,.$$

In particular, we obtain $\Omega(\mathcal{X}_t)^- = \langle \omega_1^{\mathcal{X}}, \omega_2^{\mathcal{X}} \rangle$ and $\Omega(\mathcal{X}_t)^+ = \langle \omega_3^{\mathcal{X}} \rangle$ as $\rho^{\mathcal{X}}$-eigenspaces.

(2) The space of holomorphic 1-forms on each fibre $\mathcal{Y}_t$ of the C_6-family is generated by the $(\alpha^{\mathcal{Y}})^*$-eigenforms

$$\omega_1^{\mathcal{Y}} = \frac{\mathrm{d}x}{y}\,, \qquad \omega_2^{\mathcal{Y}} = \frac{y\mathrm{d}x}{x(x-1)(x-t)}\,, \qquad \omega_3^{\mathcal{Y}} = \frac{y^4\mathrm{d}x}{x^2(x-1)^2(x-t)^2}\,.$$

In particular, we obtain $\Omega(\mathcal{Y}_t)^- = \langle \omega_1^{\mathcal{Y}}, \omega_2^{\mathcal{Y}} \rangle$ and $\Omega(\mathcal{Y}_t)^+ = \langle \omega_3^{\mathcal{Y}} \rangle$ as $\rho^{\mathcal{Y}}$-eigenspaces.

Proof. By writing their local expressions, one can check that all these forms are holomorphic. The action of ρ can be checked in the affine coordinates. □

By analysing the zeroes one obtains the following lemma.

Lemma 3.4. *Let $t \in \mathbb{P}^*$.*

(1) The forms in $\mathbb{P}\Omega(\mathcal{X}_t)^-$ having a 4-fold zero at a fixed point of $\rho^{\mathcal{X}}$ are
- *$\omega_1^{\mathcal{X}}$ which has a zero at the preimage of ∞,*
- *$\omega_2^{\mathcal{X}}$ which has a zero at the preimage of 0,*
- *$-\omega_1^{\mathcal{X}} + \omega_2^{\mathcal{X}}$ which has a zero at the preimage of 1, and*
- *$-t\omega_1^{\mathcal{X}} + \omega_2^{\mathcal{X}}$ which has a zero at the preimage of t.*

They all form an orbit under $\mathrm{Aut}(\mathcal{X}_t)$.

(2) For $t \neq 1/2$, the only form in $\mathbb{P}\Omega(\mathcal{Y}_t)^-$ which has a 4-fold zero at a fixed point of $\rho^{\mathcal{Y}}$ is $\omega_2^{\mathcal{Y}}$.

Proof. 1. For any $\mathcal{X}_t$, the preimages of 0, 1, t and ∞ are the only fixed points of $\rho^{\mathcal{X}}$. Using local charts, it is easy to see that these are the only forms with 4-fold zeroes at those points.

The last statement follows from the fact that $\mathrm{Aut}(\mathcal{X}_t)$ permutes the preimages of 0, 1, t and ∞.

2. Observe that the differential $\mathrm{d}x$ does not vanish on $\mathcal{Y}_t$ away from the preimages of 0, 1, t and ∞. The local expression around the preimages of 0 and 1 is $\mathrm{d}x = 3z^2\mathrm{d}z$ and around the preimages of t is $\mathrm{d}x = 2z\mathrm{d}z$. Looking at the local expressions, one can see that $\omega_1^{\mathcal{Y}}$ has simple zeroes at the (four) preimages of 0 and 1, and $\omega_2^{\mathcal{Y}}$ has a 4-fold zero at infinity.

Again using local charts, it is easy to see that a form $u\omega_1^{\mathcal{Y}} + v\omega_2^{\mathcal{Y}}$, $u, v \in \mathbb{C}$, can have at most 2-fold zeroes at the preimages of t.

On the other hand, if $u\omega_1^{\mathcal{Y}} + v\omega_2^{\mathcal{Y}}$ has a 4-fold zero at ∞, then the local expression above implies that $u = 0$. □

We can now state the converse of Proposition 3.1.

Proposition 3.5. *Let $t \in \mathbb{P}^*$ and let $\mathcal{O}_D$ be some quadratic discriminant.*

(1) If $\mathcal{P}(\mathcal{X}_t)$ admits proper real multiplication by $\mathcal{O}_D$ with $\omega_1^{\mathcal{X}}$ as an eigenform then $\omega_2^{\mathcal{X}}$, $-\omega_1^{\mathcal{X}}+\omega_2^{\mathcal{X}}$ and $-t\omega_1^{\mathcal{X}}+\omega_2^{\mathcal{X}}$ are also eigenforms and $(\mathcal{X}_t,\omega_1^{\mathcal{X}})$ corresponds to an orbifold point on W_D.

Moreover, if $\mathcal{X}_t \cong \mathcal{X}_{-1}$, then $(\mathcal{X}_t,\omega_1^{\mathcal{X}})$ is of order 4; if $\mathcal{X}_t \cong \mathcal{X}_{\zeta_6}$, then $(\mathcal{X}_t,\omega_1^{\mathcal{X}})$ is of order 6; otherwise, $(\mathcal{X}_t,\omega_1^{\mathcal{X}})$ is of order 2.

(2) If $\mathcal{P}(\mathcal{Y}_t)$ admits proper real multiplication by $\mathcal{O}_D$ with $\omega_2^{\mathcal{Y}}$ as an eigenform then $(\mathcal{Y}_t,\omega_2^{\mathcal{Y}})$ corresponds to an orbifold point on W_D.

Moreover, if $\mathcal{Y}_t = \mathcal{Y}_{1/2}$, then $(\mathcal{Y}_t,\omega_2^{\mathcal{Y}})$ is of order 6; otherwise, $(\mathcal{Y}_t,\omega_2^{\mathcal{Y}})$ is of order 3.

Proof. By the previous lemma, if one of the four forms in $\mathcal{X}_t$ is an eigenform for some choice of real multiplication $\mathcal{O}_D \hookrightarrow \operatorname{End}\mathcal{P}(\mathcal{X}_t)$, then the other three are also eigenforms for the choice of real multiplication conjugate by the corresponding automorphism. The statements about the points of higher order follow from Lemma 3.7 and Lemma 3.8.

The rest of the claims follows from Proposition 2.1 and Lemma 3.2. □

3.2. **Homology.** To calculate the Jacobians of the fibres of the Wollmilchsau family $\mathcal{X}$ and the C_6-family $\mathcal{Y}$, we also need a good understanding of their homology.

Consider again the general family $\mathcal{Z} \to \mathbb{P}^*$ introduced at the beginning of Section 3. Set $\mathbb{P}^*_t := \mathbb{P}^* - \{t\}$ and $\mathcal{Z}^*_t := \pi^{-1}(\mathbb{P}^*_t)$, where $\pi\colon \mathcal{Z}_t \to \mathbb{P}^1$ is the projection onto the x coordinate. We thus obtain an unramified cover and the sequence

$$1 \to \pi_1(\mathcal{Z}^*_t) \to \pi_1(\mathbb{P}^*_t) \to C_d \to 1,$$

where C_d denotes the cyclic group of order d, is exact. Let σ_i denote a simple counter-clockwise loop containing exactly the ith branch point in $\mathbb{P}^1$. Then $\pi_1(\mathbb{P}^*_t)$ is generated by $\sigma_0,\sigma_1,\sigma_t$ and σ_∞ and their product is trivial. Observe that σ_i is mapped to an element of order ${}^{d}/_{\gcd(a_i,d)}$ in C_d. Moreover, cycles in $\pi_1(\mathbb{P}^*_t)$ whose image in C_d is trivial lift to cycles in $\mathrm{H}_1(\mathcal{Z}_t,\mathbb{Z})$.

For cycles $F,G \in \mathrm{H}_1(\mathcal{Z}_t,\mathbb{Z})$, we pick representatives intersecting at most transversely and define the intersection number $F\cdot G := \sum F_p \cdot G_p$, where the sum is taken over all $p \in F\cap G$ and for any such p, we define $F_p\cdot G_p := +1$ if G approaches F "from the right in the direction of travel" and $F_p \cdot G_p := -1$ otherwise, see Figure 2.

In the following, we identify $\operatorname{Gal}(\mathcal{Z}_t/\mathbb{P}^1) = C_d$ with the dth complex roots of unity and choose the generator α as $\exp(2\pi\mathrm{i}/d)$. Since all the fibres are topologically equivalent, let us suppose for simplicity $t\in\mathbb{R}$, $t>1$. Then, the simply-connected set $\mathbb{P}^1 - [0,\infty]$ contains no ramification points and therefore has d disjoint preimages $S_1,\dots,S_d$, which we call *sheets* of $\mathcal{Z}_t$. These are permuted transitively by α and we choose the numbering so that $\alpha(S_{[n]}) = S_{[n+1]}$, where $[n] := n \bmod d$. The sheet changes are given by the monodromy: a path travelling around 0 in a counter-clockwise direction on sheet n continues onto sheet $n+a_0 \bmod d$ after crossing the interval $(0,1)$ and similarly for the other branch points.

We are now in a position to explicitly describe the fibrewise homology of $\mathcal{X}$ and $\mathcal{Y}$.

Let $F^{\mathcal{X}}$ denote the lift of $\sigma_1^{-1}\sigma_0$ that starts on sheet number 1 of $\mathcal{X}_t$ and let $G^{\mathcal{X}}$ denote the lift of $\sigma_t^{-1}\sigma_1$ that also starts on sheet 1 (see Figure 2). Observe that $F^{\mathcal{X}}\cdot G^{\mathcal{X}} = +1$.

Similarly, denote by $F^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$ the lifts of $\sigma_1^{-1}\sigma_0$ and $\sigma_\infty^{-3}\sigma_t$, that start on sheet 1 and 5 of $\mathcal{Y}_t$, respectively (see Figure 3). Observe that $F^{\mathcal{Y}}\cdot G^{\mathcal{Y}} = 0$.

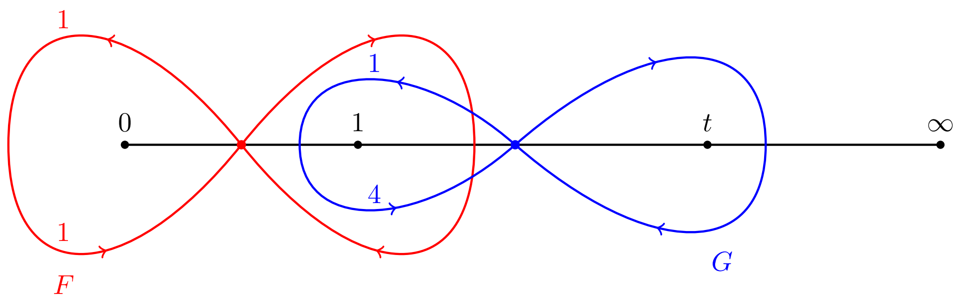

FIGURE 2. The cycles $F^{\mathcal{X}}$ and $G^{\mathcal{X}}$ on $\mathcal{X}_t$. The upper-left parts of both cycles lie on sheet number 1. Observe that $F^{\mathcal{X}} \cdot G^{\mathcal{X}} = 1$.

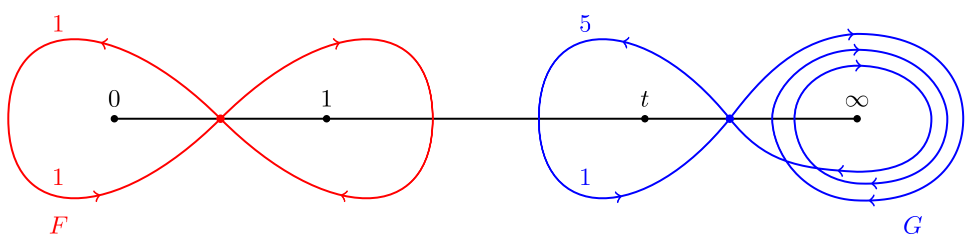

FIGURE 3. The cycles $F^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$ on $\mathcal{Y}_t$. The upper-left parts of them lie on sheets number 1 and 5, respectively.

To ease notation, we will drop superscripts in the following lemma, as no confusion can arise.

Lemma 3.6. *Let $t \in \mathbb{P}^*$.*

(1) The cycles $F, \alpha F, \alpha^2 F, G, \alpha G, \alpha^2 G$ yield a basis of $\mathrm{H}_1(\mathcal{X}_t, \mathbb{Z})$. Moreover, the cycles

$$F + \alpha F + G + \alpha G,\ -G + \alpha^2 G,\ \alpha F + \alpha^2 F - G + \alpha^2 G,\ F + 2\alpha F + \alpha^2 F$$

span a $(1,2)$-polarised, ρ-anti-invariant sublattice of $\mathrm{H}_1(\mathcal{X}_t, \mathbb{Z})$, which we denote by $\mathrm{H}_1^-(\mathcal{X}_t, \mathbb{Z})$. The complementary ρ-invariant sublattice, $\mathrm{H}_1^+(\mathcal{X}_t, \mathbb{Z})$, is spanned by $F + \alpha^2 F, G + \alpha^2 G$.

(2) The cycles $F, \alpha F, \alpha^3 F, \alpha^4 F, G, \alpha G$ yield a basis of $\mathrm{H}_1(\mathcal{Y}_t, \mathbb{Z})$. Moreover, the cycles

$$F - \alpha^3 F,\ \alpha^4 F - \alpha F,\ G,\ \alpha G$$

span a $(1,2)$-polarised, ρ-anti-invariant sublattice of $\mathrm{H}_1(\mathcal{Y}_t, \mathbb{Z})$, which we denote by $\mathrm{H}_1^-(\mathcal{Y}_t, \mathbb{Z})$. The complementary ρ-invariant sublattice, $\mathrm{H}_1^+(\mathcal{Y}_t, \mathbb{Z})$, is spanned by $F + \alpha^3 F, \alpha^4 F + \alpha F$.

Proof. 1. An elementary but somewhat tedious calculation yields the intersection matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 0 \end{pmatrix}$$

for the above cycles on $\mathcal{X}_t$. As it has rank 6 and determinant 1, these cycles span all of $\mathrm{H}_1(\mathcal{X}_t, \mathbb{Z})$. Furthermore, this immediately provides us with the relations

$$\alpha^3 F = -F - \alpha F - \alpha^2 F \quad \text{and} \quad \alpha^3 G = -G - \alpha G - \alpha^2 G,$$

which confirms the claimed anti-invariance. The change to the second set of cycles yields

$$\begin{pmatrix} 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 2 & & \\ -1 & 0 & 0 & 0 & & \\ 0 & -2 & 0 & 0 & & \\ & & & & 0 & 2 \\ & & & & -2 & 0 \end{pmatrix}$$

where the upper-left block is the anti-invariant and the lower-right block is the invariant part. Calculating determinants, we see that both blocks have determinant 4, proving the claim about the polarisation.

2. Proceeding as before, one finds the following intersection matrix for the cycles on $\mathcal{Y}_t$

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

proving that they generate $\mathrm{H}_1(\mathcal{Y}_t, \mathbb{Z})$, and the following one for the second set of cycles

$$\begin{pmatrix} 0 & 2 & 0 & 0 & & \\ -2 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 1 & & \\ 0 & 0 & -1 & 0 & & \\ & & & & 0 & 2 \\ & & & & -2 & 0 \end{pmatrix},$$

yielding the $(1,2) \times (2)$-polarisation on the product. □

3.3. **Special points.** We briefly summarise some of the subtleties occurring at those points admitting additional symmetries.

The curve $\mathcal{X}_2$. In the Wollmilchsau family $\mathcal{X}$, the fibres over $1/2$, -1 and 2 form an orbit under the action of $\mathfrak{S}_3$. Over these points, $\alpha^{\mathcal{X}}$ extends to an automorphism $\beta^{\mathcal{X}}$ satisfying $(\beta^{\mathcal{X}})^2 = \alpha^{\mathcal{X}}$, i.e. a symmetry of order 8, making them all isomorphic to the well-known *Fermat curve*. More precisely, $\beta^{\mathcal{X}}$ may be obtained by lifting the autmorphism that permutes two of the branch points and fixes the remaining pair on $\mathbb{P}^1$. Note that this may be achieved in two ways, e.g. for $t = 2$, we obtain

$$\beta_1^{\mathcal{X}} : (x, y) \mapsto \left(\frac{x}{x-1}, \zeta_8 \frac{y}{x-1} \right) \text{ and}$$
$$\beta_2^{\mathcal{X}} : (x, y) \mapsto (2 - x, \zeta_8 y).$$

Observe that $\beta_1^{\mathcal{X}}$ fixes 2 and 0 while interchanging 1 and ∞, while $\beta_2^{\mathcal{X}}$ fixes 1 and ∞ while interchanging 2 and 0. It is straight-forward to check the analogous statement of Lemma 3.4 in this case.

Lemma 3.7. *Let t be one of $1/2$, -1 or 2. Then the two forms from Lemma 3.4 with zeros at the fixed points of $\beta_1^{\mathcal{X}}$ are eigenforms for $\beta_1^{\mathcal{X}}$, while the other two forms are eigenforms for $\beta_2^{\mathcal{X}}$.*

The curve $\mathcal{Y}_{1/2}$ (or $\mathcal{X}_{\zeta_6}$). The only member of the C_6-family $\mathcal{Y}$ whose automorphism group contains a cyclic group of order larger than 6 is $\mathcal{Y}_{1/2}$, admitting an automorphism of order 12, $\beta^{\mathcal{Y}}(x, y) = (1 - x, \zeta_{12}^7 y)$, satisfying $(\beta^{\mathcal{Y}})^2 = \alpha^{\mathcal{Y}}$. In contrast to the case of $\mathcal{X}_2$, however, the automorphism $\beta^{\mathcal{Y}}$ is unique.

Recall that, by Proposition 3.1, the curve $\mathcal{Y}_{1/2}$ is isomorphic to the curve $\mathcal{X}_{\zeta_6}$ of the Wollmilchsau family. However, here we will use the model of the curve as a member of the C_6-family.

Note first that $\beta^{\mathcal{Y}}$ descends to the automorphism $z \mapsto 1 - z$ of $\mathbb{P}^1$. Moreover $\beta^{\mathcal{Y}}$ fixes ∞ with rotation number ζ_{12} and therefore $\beta^{\mathcal{Y}}$ acts as $(1^+, 1^-, 2^+, \ldots, 6^+, 6^-)$ on the half-sheets, where we write k^+ (respectively k^-) for the upper half-plane (respectively lower half-plane) corresponding to the kth sheet.

By letting the initial points of $F^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$ go to 1 and ∞, respectively, and shrinking the cycles around the preimages of 0, 1, t and ∞ one can use the (equivalent) choice of cycles pictured in Figure 4.

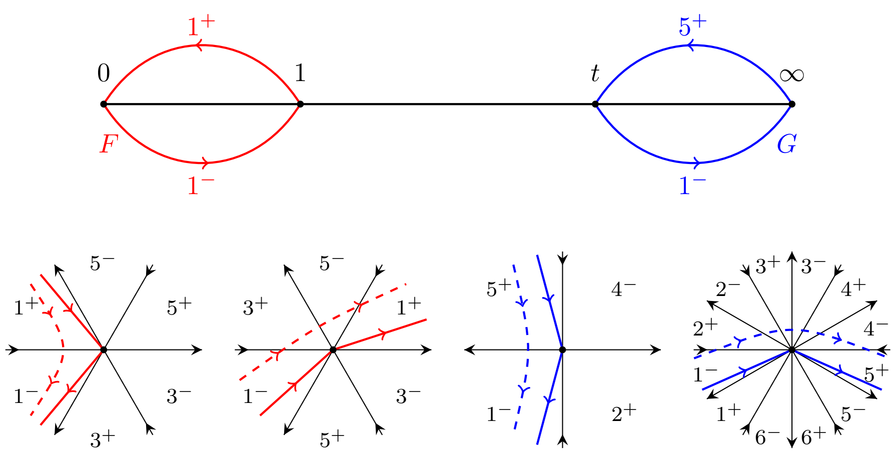

FIGURE 4. The shrunk cycles $F^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$, and the process of shrinking around the preimages of 0, 1, t and ∞, respectively.

After the shrinking process, the cycles $F^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$ in $\mathcal{Y}_{1/2}$ have the shape depicted in Figure 5.

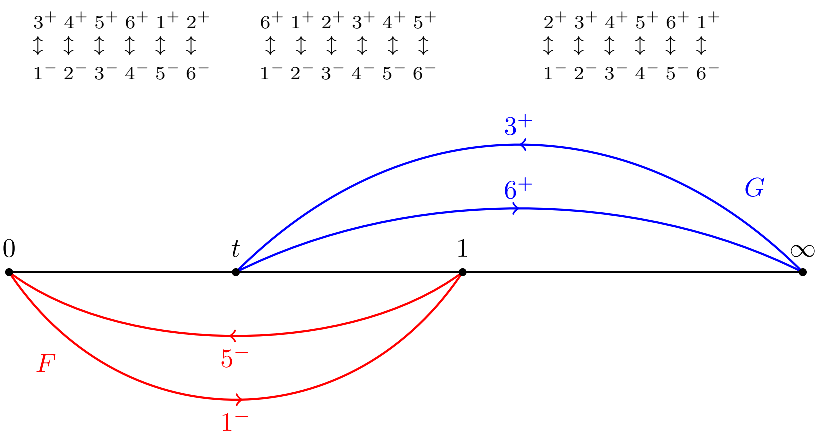

FIGURE 5. The cycles $F^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$ in $\mathcal{Y}_{1/2}$.

Taking all this into account, one can easily calculate the analytic and rational representations of $\beta^{\mathcal{Y}}$.

Lemma 3.8. *The analytic and rational representations of $\beta^{\mathcal{Y}}$ with respect to the bases $\mathrm{H}_1(\mathcal{Y}_{1/2}, \mathbb{Z}) = \langle F^{\mathcal{Y}}, \alpha^{\mathcal{Y}} F^{\mathcal{Y}}, (\alpha^{\mathcal{Y}})^3 F^{\mathcal{Y}}, (\alpha^{\mathcal{Y}})^4 F^{\mathcal{Y}}, G^{\mathcal{Y}}, \alpha^{\mathcal{Y}} G^{\mathcal{Y}} \rangle_{\mathbb{Z}}$ and $\Omega(\mathcal{Y}_{1/2}) = \langle \omega_1^{\mathcal{Y}}, \omega_2^{\mathcal{Y}}, \omega_3^{\mathcal{Y}} \rangle$ are given, respectively, by*

$$A_{\beta^{\mathcal{Y}}} = \begin{pmatrix} \zeta_{12}^{-1} & 0 & 0 \\ 0 & \zeta_{12}^{7} & 0 \\ 0 & 0 & \zeta_{12}^{-2} \end{pmatrix} \qquad R_{\beta^{\mathcal{Y}}} = \begin{pmatrix} 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 2 \end{pmatrix}.$$

In particular, $\omega_2^{\mathcal{Y}}$ is an eigenform for $\beta^{\mathcal{Y}}$.

3.4. **Stable reduction of degenerate fibres.** While the C_6-family is not compact in $\mathcal{M}_3$, it turns out that all fibres of its closure in $\overline{\mathcal{M}_3}$, the Deligne-Mumford compactification, admit compact Jacobians, i.e. that the Torelli image of $\overline{\mathcal{Y}}$ is contained in $\mathcal{A}_3$. Moreover, this analysis will be invaluable when constructing a fundamental domain for $\mathcal{Y}$ later.

The degenerate fibres of $\mathcal{Y}$. The degenerate fibres of the C_6-family $\mathcal{Y}$ correspond to $t = 0, 1, \infty$. To describe them, we resort to the theory of *admissible covers.* For a brief overview of the tools needed in this special case, see e.g. [BM10, §4.1] and the references therein.

The stable reduction when $t \to 1$ (equivalently, when $t \to 0$) yields the two components

$$\overline{\mathcal{Y}}_1^1 : y^6 = x^2(x-1)^5\,, \quad \text{of genus } 2,$$
$$\overline{\mathcal{Y}}_1^2 : y^6 = x^2(x-1)^3\,, \quad \text{of genus } 1.$$

The stable reduction when $t \to \infty$ yields the three components

$$\overline{\mathcal{Y}}_\infty^1 : y^6 = x^2(x-1)^2\,, \quad \text{consisting of two components of genus 1,}$$
$$\overline{\mathcal{Y}}_\infty^2 : y^6 = x^3(x-1)^5\,, \quad \text{of genus 1.}$$

A simple calculation gives the following lemma.

Lemma 3.9. *The degeneration of the $(\alpha^{\mathcal{Y}})^*$-eigenforms of Lemma 3.3 for $t \to 1$ is given by*

$$\omega_1^1 = \frac{\mathrm{d}x}{y} \text{ in } \overline{\mathcal{Y}}_1^1\,, \qquad \omega_2^1 = \frac{y\mathrm{d}x}{x(x-1)} \text{ in } \overline{\mathcal{Y}}_1^2\,, \qquad \omega_3^1 = \frac{y^4\mathrm{d}x}{x^2(x-1)^4} \text{ in } \overline{\mathcal{Y}}_1^1\,,$$

and for $t \to \infty$ by

$$\omega_1^\infty = \frac{\mathrm{d}x}{y} \text{ in } \overline{\mathcal{Y}}_\infty^2\,, \qquad \omega_2^\infty = \frac{y\mathrm{d}x}{x(x-1)} \text{ in } \overline{\mathcal{Y}}_\infty^1\,, \qquad \omega_3^\infty = \frac{\mathrm{d}x}{y^2} \text{ in } \overline{\mathcal{Y}}_\infty^1\,,$$

where the differentials are identically zero in the components where they are not defined.

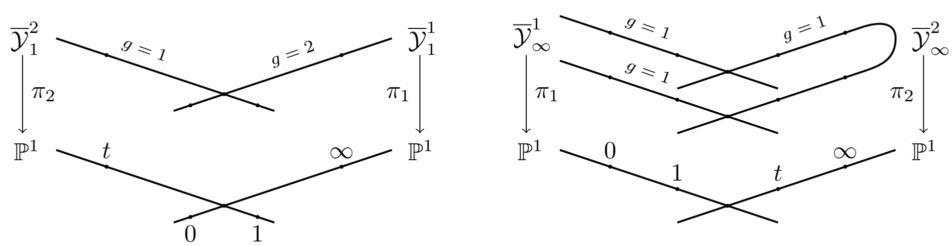

FIGURE 6. The stable fibres $\overline{\mathcal{Y}}_1$ and $\overline{\mathcal{Y}}_\infty$.

Via the shrinking process introduced above, one can compute the degeneration of the cycles in both cases (see Figure 7). In the following lemma, we sum up some results about the homology of the degenerate fibres that we will need later.

Lemma 3.10. *Let F^∞, G^∞ and F^1, G^1 denote the cycles on $\overline{\mathcal{Y}}_\infty$ and $\overline{\mathcal{Y}}_1$ corresponding to the degeneration of $F^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$.*

(1) F^∞ and G^∞ live in $\overline{\mathcal{Y}}_\infty^1$ and $\overline{\mathcal{Y}}_\infty^2$ respectively.
(2) There is a decomposition of cycles $F^1 = F_1^1 + F_2^1$ and $G^1 = G_1^1 + G_2^1$, where F_k^1, G_k^1 are cycles in the component $\overline{\mathcal{Y}}_1^k$ going through the nodal point.
Moreover, one has the following intersection matrices for the sets of cycles $\{F_k^1, \alpha^{\mathcal{Y}} F_k^1, (\alpha^{\mathcal{Y}})^3 F_k^1, (\alpha^{\mathcal{Y}})^4 F_k^1, G_k^1, \alpha^{\mathcal{Y}} G_k^1\}$, for $k = 1, 2$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \\ -1 & -1 & 1 & 1 & 0 & 2 \\ 0 & -1 & 0 & 1 & -2 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \end{pmatrix},$$

respectively.

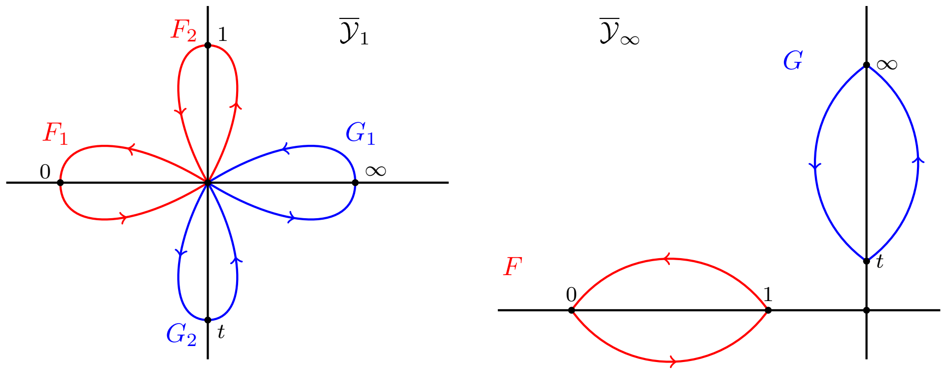

FIGURE 7. The bases of homology in $\overline{\mathcal{Y}}_1$ and $\overline{\mathcal{Y}}_\infty$ as lifts of cycles in $\mathbb{P}^1$ by π_1 and π_2.

Proof. In the case of $\overline{\mathcal{Y}}_\infty$, it is obvious from Figure 3 that the degeneration of the cycles $F^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$ lie in $\overline{\mathcal{Y}}^1_\infty$ and $\overline{\mathcal{Y}}^2_\infty$ respectively.

The case $\overline{\mathcal{Y}}_1$ is more delicate. It follows again from Figure 3 that the degeneration of both $F^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$ are the union of cycles in $\overline{\mathcal{Y}}^1_1$ and $\overline{\mathcal{Y}}^2_1$ meeting at the nodal point. In fact, since the points in $\overline{\mathcal{Y}}_1$ corresponding to the preimages of 0 and 1 (respectively t and ∞) lie in different components, it is clear that F^1 (respectively G^1) will decompose as the sum $F^1_1 + F^1_2$ (respectively $G^1_1 + G^1_2$) of cycles in $\overline{\mathcal{Y}}^1_1$ and $\overline{\mathcal{Y}}^2_1$.

Consider first the component $\overline{\mathcal{Y}}^2_1$, isomorphic to $y^6 = x^2(x-1)^3$. Note that the preimages of 0 and 1 under π_2 correspond to the preimages of 1 and t in the general member of our family $\mathcal{Y}_t$. Let us denote by $Q \in \overline{\mathcal{Y}}^2_1$ the nodal point and suppose, for simplicity, that its image $q \in \mathbb{P}^1$ under π_2 lies in the interval $[1, 0]$. Removing this interval and proceeding as before we get the picture in Figure 8, where the sheet changes follow from studying the behaviour of $F^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$ around the preimages of 1 and t in the general member of our family (see Figure 4).

One can get a similar picture for the other component $\overline{\mathcal{Y}}^1_1$. Now a tedious but straightforward calculation yields the intersection matrices. □

4. THE PRYM-TORELLI IMAGES

To understand the orbifold points of W_D, by Proposition 3.5, we must determine which $\mathcal{X}_t$ and $\mathcal{Y}_t$ admit real multiplication. Therefore, the aim of this section is to concretely calculate the period matrices of the families of Prym varieties $\mathcal{P}(\mathcal{X}_t)$ of the Wollmilchsau family and $\mathcal{P}(\mathcal{Y}_t)$ of the C_6-family.

4.1. The Prym variety $\mathcal{P}(\mathcal{X}_t)$.

In the case of the Wollmilchsau family $\mathcal{X}$, all the fibres $\mathcal{X}_t$ are sent to the same Prym variety by the Prym-Torelli map.

Proposition 4.1. *For all $t \in \mathbb{P}^*$, the Prym variety $\mathcal{P}(\mathcal{X}_t)$ is isomorphic to $\mathbb{C}^2/\Lambda$, where $\Lambda = \mathcal{P}\Pi^{\mathcal{X}} \cdot \mathbb{Z}^4$ for*

$$\mathcal{P}\Pi^{\mathcal{X}} = \begin{pmatrix} \frac{-1-\mathrm{i}}{2} & 1 & 1 & 0 \\ 1 & -1-\mathrm{i} & 0 & 2 \end{pmatrix},$$

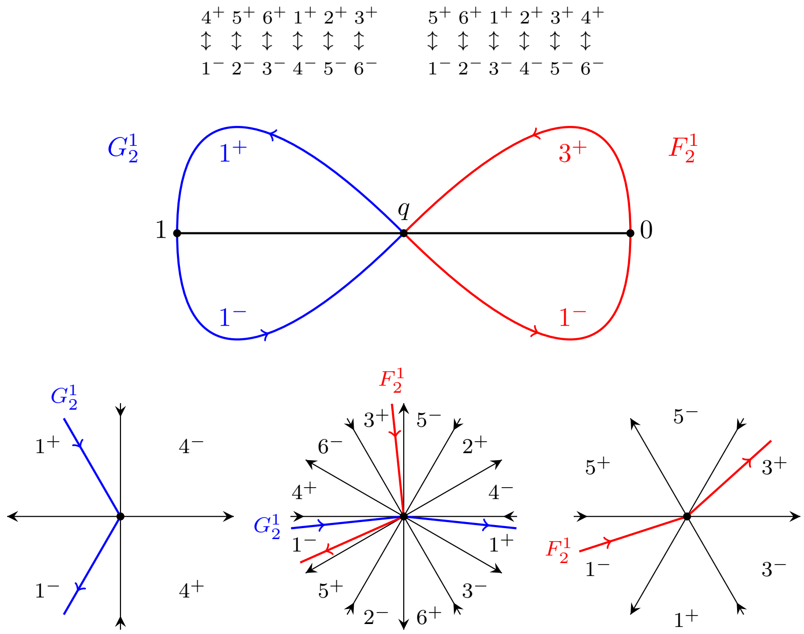

FIGURE 8. Degenerate cycles on $\overline{\mathcal{Y}}_1^2$ and their behaviour around the preimages of t, q and 1, respectively. Note that $F_2^1 \cdot G_2^1 = -1$.

together with the polarisation induced by the intersection matrix

$$E^{\mathcal{X}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}.$$

In particular, the image of the Wollmilchsau family $\mathcal{X}$ in the Prym locus is a single point.

Calculating Jacobians of curves with automorphisms can be done by a method attributed to Bolza, see [BL04, Chap. 11.7] for details. The idea is to determine the analytic and rational representations of the automorphisms and use this information to find relations in the period matrix.

The group of automorphisms of a general member of the Wollmilchsau family $\mathcal{X}$ is generated by $\alpha \coloneqq \alpha^{\mathcal{X}}$ and the involutions

$$\gamma\colon (x, y) \mapsto \left(\frac{t}{x}, \frac{y\sqrt{t}}{x}\right) \quad \text{and} \quad \delta\colon (x, y) \mapsto \left(\frac{t(x-1)}{x-t}, \frac{-y\sqrt{t(t-1)}}{x-t}\right).$$

Note that γ and δ are lifts by π of the automorphisms of $\mathbb{P}^1$ given by $z \mapsto \frac{t}{z}$ and $z \mapsto \frac{tz-t}{z-t}$ respectively. In particular, these two involutions generate a Klein four-group acting on the fixed points of $\rho^{\mathcal{X}}$.

By Lemma 3.3, the action of α^* on $\Omega(\mathcal{X}_t)$ is given by

$$\begin{pmatrix} \mathrm{i} & 0 & 0 \\ 0 & \mathrm{i} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

in the eigenform basis. The automorphisms γ and δ induce analytic representations

$$\gamma^* = \begin{pmatrix} 0 & -\sqrt{t} & 0 \\ -\dfrac{1}{\sqrt{t}} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \delta^* = \begin{pmatrix} \dfrac{-t}{\sqrt{t(t-1)}} & \dfrac{-t}{\sqrt{t(t-1)}} & 0 \\ \dfrac{1}{\sqrt{t(t-1)}} & \dfrac{t}{\sqrt{t(t-1)}} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

To calculate the rational representation, let us suppose again $t \in \mathbb{R}$, $t > 1$. Keeping track of the action of γ and δ on the branching points of π and on the half-sheets of the cover, one can write down the action of these automorphisms in the homology

$$\gamma F^{\mathcal{X}} = -\alpha^2 F^{\mathcal{X}} + G^{\mathcal{X}} + \alpha G^{\mathcal{X}}, \qquad \gamma G^{\mathcal{X}} = -G^{\mathcal{X}},$$
$$\delta F^{\mathcal{X}} = -F^{\mathcal{X}}, \qquad \delta G^{\mathcal{X}} = -\alpha F^{\mathcal{X}} - \alpha^2 F^{\mathcal{X}} - \alpha^2 G^{\mathcal{X}}.$$

Remark 4.2. *Observe that γ and δ act as involutions and the quotient is $\mathcal{X}_t/\gamma \cong \mathcal{X}_t/\delta \cong E_{\mathrm{i}}$, where E_{i} is the unique elliptic curve with an order four automorphism. Indeed, $\mathcal{X}_t$ is not hyperelliptic and δ and γ have fixed points (e.g. preimages of $\sqrt{t}$ and $t-\sqrt{t(t-1)}$ on $\mathcal{X}_t$), therefore the quotient has genus 1. Moreover, α commutes with both δ and γ, hence descends to an order four automorphism of the quotient elliptic curve.*

Proof of Proposition 4.1. To calculate the Jacobian $\mathrm{Jac}(\mathcal{X})$ write $f_i := f_i^{\mathcal{X}}(t) = \int_{F^{\mathcal{X}}} \omega_i^{\mathcal{X}}$ and $g_i := g_i(t) = \int_{G^{\mathcal{X}}} \omega_i^{\mathcal{X}}$. From the action of α one can deduce that the Jacobian of $\mathcal{X}_t$ in the bases of Lemmas 3.3 and 3.6 is given by the period matrix

$$\Pi_t^{\mathcal{X}} = \begin{pmatrix} f_1 & \mathrm{i}f_1 & -f_1 & g_1 & \mathrm{i}g_1 & -g_1 \\ f_2 & \mathrm{i}f_2 & -f_2 & g_2 & \mathrm{i}g_2 & -g_2 \\ f_3 & -f_3 & f_3 & g_3 & -g_3 & g_3 \end{pmatrix}.$$

Using the actions of γ and δ both on $\Omega(\mathcal{X}_t)$ and $\mathrm{H}_1(\mathcal{X}_t, \mathbb{Z})$ one gets the relations

$$f_1 = -\sqrt{t} f_2 - g_1 - \mathrm{i}g_1, \quad g_2 = \frac{g_1}{\sqrt{t}}, \quad g_1 = \frac{-f_2\sqrt{t}(1-\sqrt{t}+\sqrt{t-1})}{(1+\mathrm{i})(\sqrt{t-1}-\sqrt{t})}.$$

By changing to the basis of $\mathrm{H}_1^-(\mathcal{X}_t, \mathbb{Z}) \oplus \mathrm{H}_1^+(\mathcal{X}_t, \mathbb{Z})$ given in Lemma 3.6 one gets

$$\begin{pmatrix} (1+\mathrm{i})(f_1+g_1) & -2g_1 & -2g_1+(\mathrm{i}-1)f_1 & 2\mathrm{i}f_1 & 0 & 0 \\ (1+\mathrm{i})(f_2+g_2) & -2g_2 & -2g_2+(\mathrm{i}-1)f_2 & 2\mathrm{i}f_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2f_3 & 2g_3 \end{pmatrix}$$

and sees that the Jacobian $\mathrm{Jac}(\mathcal{X}_t)$ is isogenous to the product $\mathcal{P}(\mathcal{X}_t) \times \mathrm{Jac}(\mathcal{X}_t/\rho^{\mathcal{X}})$, where $\mathcal{P}(\mathcal{X}_t)$ is $(1,2)$-polarised and $\mathrm{Jac}(\mathcal{X}_t/\rho^{\mathcal{X}})$ is (2)-polarised. Note that the polarisation on $\mathcal{P}(\mathcal{X}_t)$ is given by the principal 4×4 minor in the intersection matrix in the proof of Lemma 3.6, which agrees with $E^{\mathcal{X}}$.

Finally, we can change the basis of $\Omega(\mathcal{X}_t)^-$ by the matrix

$$Q_t = \frac{1}{\sqrt{t-1}f_2}\begin{pmatrix} \frac{(-1-\mathrm{i})(\sqrt{t}-\sqrt{t-1})}{4\sqrt{t}} & \frac{-1-i}{4} \\ \frac{\mathrm{i}}{2\sqrt{t}} & \frac{i(\sqrt{t}-\sqrt{t-1})}{2} \end{pmatrix}, \tag{2}$$

to get the period matrix

$$\begin{pmatrix} \mathcal{P}\Pi^{\mathcal{X}} & 0 \\ 0 & \mathcal{E}\Pi_t^{\mathcal{X}} \end{pmatrix} \text{ where } \mathcal{P}\Pi^{\mathcal{X}} := \begin{pmatrix} \frac{-1-\mathrm{i}}{2} & 1 & 1 & 0 \\ 1 & -1-\mathrm{i} & 0 & 2 \end{pmatrix} \text{ and } \mathcal{E}\Pi_t^{\mathcal{X}} := \begin{pmatrix} 2f_3 & 2g_3 \end{pmatrix}.$$

Note that $\mathcal{P}\Pi^{\mathcal{X}}$ no longer depends on t, proving the final statement. □

Remark 4.3. *These results are equivalent to those of Guàrdia in [Guà01]. However, we cannot simply apply his results for two reasons. First, we are not restricted to real branching values and in particular the curve $\mathcal{X}_{\zeta_6}$ plays a special role. More importantly, in order to study the points of intersection with the Prym-Teichmüller curves W_D, we need to keep track of the differential forms with a 4-fold zero in each fibre of the family. As a consequence, we need an explicit expression of the elements of $\Omega(\mathcal{X}_t)^-(4)$ in the basis in which the period matrix $\mathcal{P}\Pi^{\mathcal{X}}$ above is written.*

The endomorphism ring $\operatorname{End}\mathcal{P}(\mathcal{X}_t)$**.** To see when $\mathcal{P}(\mathcal{X}_t)$ has real multiplication, we need a good understanding of the endomorphism ring.

Proposition 4.4. *The endomorphism algebra $\operatorname{End}_{\mathbb{Q}}\mathcal{P}(\mathcal{X}_t)$ is the algebra isomorphic to $M_2(\mathbb{Q}[\mathrm{i}])$ generated by the identity and the automorphisms α, γ, δ and $\gamma\delta$.*

Proof. Note that the automorphisms α, γ and δ of $\mathcal{X}_t$ preserve the spaces $\Omega(\mathcal{X}_t)^-$ and $\mathrm{H}_1^-(\mathcal{X}_t,\mathbb{Z})$, so they induce automorphisms of the Prym variety. One can construct their analytic and rational representations in the bases of Lemmas 3.3 and 3.6 to obtain

$$A_\alpha = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \qquad R_\alpha = \begin{pmatrix} 1 & -2 & -2 & 0 \\ -1 & 1 & 0 & -2 \\ 2 & -2 & -1 & 2 \\ -1 & 2 & 1 & -1 \end{pmatrix};$$

$$A_\gamma = \begin{pmatrix} 0 & \frac{1-\mathrm{i}}{2} \\ 1+\mathrm{i} & 0 \end{pmatrix}, \quad R_\gamma = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$A_\delta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad R_\delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}.$$

Since A_α lies in the centre of $M_2(\mathbb{C})$ and the involutions γ and δ anti-commute, the endomorphism algebra $\operatorname{End}_{\mathbb{Q}}\mathcal{P}(\mathcal{X}_t)$ must contain the (definite) quaternion algebra $F = \langle A_\alpha, A_\gamma, A_\delta\rangle_{\mathbb{Q}} \cong M_2(\mathbb{Q}[\mathrm{i}])$. It is easy to see that this already has to be the entire algebra $\operatorname{End}_{\mathbb{Q}}\mathcal{P}(\mathcal{X}_t)$ (see [BL04, Prop. 13.4.1]). In particular any element of $\operatorname{End}_{\mathbb{Q}}\mathcal{P}(\mathcal{X}_t)$ can be written as a $\mathbb{Q}[\mathrm{i}]$-linear combination of Id, A_γ, A_δ and $A_{\gamma\delta}$. □

Recall that for any polarised abelian variety the Rosati inovolution $\cdot'$ on the endomorphism ring is induced by the polarisation. Therefore, given an element

$\varphi \in \operatorname{End}_{\mathbb{Q}} \mathcal{P}(\mathcal{X}_t)$ with rational representation R_φ, its image φ' under the Rosati involution has rational representation $E^{-1} R_\varphi^\top E$, where $E = E^{\mathcal{X}}$ is the polarisation matrix from above. It is then easy to check that $\alpha' = -\alpha$, $\gamma' = \gamma$, $\delta' = \delta$ and $(\gamma\delta)' = -\gamma\delta$. Under the embedding $F \hookrightarrow M_2(\mathbb{C})$ given by the analytic representation, the Rosati involution is the restriction of the involution

$$\begin{array}{ccc} M_2(\mathbb{C}) & \to & M_2(\mathbb{C}) \\ B & \mapsto & A^{-1} B^H A \end{array}, \quad \text{for } A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

where B^H denotes the hermitian transpose.

This gives us a simple criterion to check whether a specific rational endomorphism actually lies in $\operatorname{End} \mathcal{P}(\mathcal{X}_t)$.

4.2. **The Prym variety $\mathcal{P}(\mathcal{Y}_t)$.** In the case of the C_6-family $\mathcal{Y}$, we have the following characterisation.

Proposition 4.5. *For all $t \in \mathbb{P}^*$, the Prym variety $\mathcal{P}(\mathcal{Y}_t) = \mathbb{C}^2/\Lambda_t$, where $\Lambda_t = \mathcal{P}\Pi_t^{\mathcal{Y}} \cdot \mathbb{Z}^4$ for*

$$\mathcal{P}\Pi_t^{\mathcal{Y}} = \begin{pmatrix} 2f & 2\zeta_6^2 f & 1 & \zeta_6^{-1} \\ 2 & 2\zeta_6^{-2} & 2f & 2\zeta_6 f \end{pmatrix},$$

where $f := f^{\mathcal{Y}}(t) = \int_{F^{\mathcal{Y}}} \omega_1^{\mathcal{Y}}$, together with the polarisation induced by the intersection matrix

$$E^{\mathcal{Y}} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

As above, we use Bolza's method for calculating the period matrix. Fortunately, in this case it suffices to regard $\alpha := \alpha^{\mathcal{Y}}$.

By Lemma 3.3, the action of α^* on $\Omega(\mathcal{Y}_t)$ is given by

$$\begin{pmatrix} \zeta_6^{-1} & 0 & 0 \\ 0 & \zeta_6 & 0 \\ 0 & 0 & \zeta_6^4 \end{pmatrix}$$

in the eigenform basis.

Proof of Proposition 4.5. Again, we write $f_i := f_i^{\mathcal{Y}}(t) = \int_{F^{\mathcal{Y}}} \omega_i^{\mathcal{Y}}$ and $g_i := g_i^{\mathcal{Y}}(t) = \int_{G^{\mathcal{Y}}} \omega_i^{\mathcal{Y}}$. Since $\alpha^3(G^{\mathcal{Y}}) = \rho^{\mathcal{Y}}(G^{\mathcal{Y}}) = -G^{\mathcal{Y}}$ and $(\rho^{\mathcal{Y}})^* \omega_3^{\mathcal{Y}} = -\omega_3^{\mathcal{Y}}$, one has $g_3 = 0$. Using the action of α on $\Omega(\mathcal{Y}_t)$, one gets that, in these bases, the period matrix of $\mathcal{Y}_t$ reads

$$\Pi_t^{\mathcal{Y}} = \begin{pmatrix} f_1 & \zeta_6^{-1} f_1 & -f_1 & \zeta_6^2 f_1 & g_1 & \zeta_6^{-1} g_1 \\ f_2 & \zeta_6 f_2 & -f_2 & \zeta_6^{-2} f_2 & g_2 & \zeta_6 g_2 \\ f_3 & \zeta_6^{-2} f_3 & f_3 & \zeta_6^{-2} f_3 & 0 & 0 \end{pmatrix}. \tag{4}$$

Moreover, by normalising $g_1 = f_2 = f_3 = 1$ and using Riemann's relations one sees

$$\Pi_t^{\mathcal{Y}} E^{-1} \left(\Pi_t^{\mathcal{Y}}\right)^T = 0 \Rightarrow g_2 = 2f_1, \text{ and}$$
$$\mathrm{i}\Pi_t^{\mathcal{Y}} E^{-1} \left(\overline{\Pi_t^{\mathcal{Y}}}\right)^T > 0 \Rightarrow 2|f_1|^2 - 1 < 0.$$

Writing $f := f_1$, we finally get

$$\Pi_t^{\mathcal{Y}} = \begin{pmatrix} f & \zeta_6^{-1} f & -f & \zeta_6^2 f & 1 & \zeta_6^{-1} \\ 1 & \zeta_6 & -1 & \zeta_6^{-2} & 2f & 2\zeta_6 f \\ 1 & \zeta_6^{-2} & 1 & \zeta_6^{-2} & 0 & 0 \end{pmatrix}. \tag{5}$$

As above, the Jacobian $\mathrm{Jac}(\mathcal{Y}_t)$ is isogenous to the variety $\mathcal{P}(\mathcal{Y}_t) \times \mathrm{Jac}(\mathcal{Y}_t/\rho^{\mathcal{Y}})$, whose period matrix is obtained by changing to the basis of $\mathrm{H}_1^-(\mathcal{Y}_t, \mathbb{Z}) \oplus \mathrm{H}_1^+(\mathcal{Y}_t, \mathbb{Z})$ of Lemma 3.6, yielding

$$\begin{pmatrix} \mathcal{P}\Pi_t^{\mathcal{Y}} & 0 \\ 0 & \mathcal{E}\Pi_t^{\mathcal{Y}} \end{pmatrix}, \text{ where } \mathcal{P}\Pi_t^{\mathcal{Y}} := \begin{pmatrix} 2f & 2\zeta_6^2 f & 1 & \zeta_6^{-1} \\ 2 & 2\zeta_6^{-2} & 2f & 2\zeta_6 f \end{pmatrix} \text{ and } \mathcal{E}\Pi_t^{\mathcal{Y}} := \begin{pmatrix} 2 & 2\zeta_6^{-2} \end{pmatrix}.$$

The polarisation on $\mathcal{P}(\mathcal{Y}_t)$ is again given by the principal 4×4 minor in the intersection matrix in the proof of Lemma 3.6, which agrees with $E^{\mathcal{Y}}$. □

The endomorphism ring $\operatorname{End} \mathcal{P}(\mathcal{Y}_t)$. In this section we study the endomorphism ring $\operatorname{End} \mathcal{P}(\mathcal{Y}_t)$ and the endomorphism algebra $\operatorname{End}_{\mathbb{Q}} \mathcal{P}(\mathcal{Y}_t)$ in order to get a description of the C_6-family $\mathcal{Y}$ as a Shimura curve. More precisely, let $\mathfrak{M}$ denote the maximal order

$$\mathfrak{M} = \mathbb{Z}\left[\frac{\mathbf{1}+\mathbf{j}}{2}, \frac{\mathbf{1}-\mathbf{j}}{2}, \frac{\mathbf{i}+\mathbf{ij}}{2}, \frac{\mathbf{i}-\mathbf{ij}}{2}\right] \tag{6}$$

in the quaternion algebra

$$F := \left\{x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{ij} \ : \ x_k \in \mathbb{Q}\,,\, \mathbf{i}^2 = 2\,,\, \mathbf{j}^2 = -3\right\} \cong \left(\frac{2,-3}{\mathbb{Q}}\right).$$

We will prove the following.

Proposition 4.6. *The Prym-Torelli map gives an isomorphism between the compactification $\overline{\mathcal{Y}}$ of the C_6-family $\mathcal{Y}$ and the Shimura curve $\mathbb{H}/\Delta(2,6,6)$, whose points correspond to abelian surfaces with a $(1,2)$ polarisation, endomorphism ring $\operatorname{End} A \cong \mathfrak{M}$ and Rosati involution given by (7).*

Let us first calculate $\operatorname{End} \mathcal{P}(\mathcal{Y}_t)$. Since the automorphism α of $\mathcal{Y}_t$ induces an automorphism of $\mathcal{P}(\mathcal{Y}_t)$ and $\mathbf{j} := 2\alpha - 1$ satisfies $\mathbf{j}^2 = -3$, there is always an embedding $\mathbb{Q}(\sqrt{-3}) \hookrightarrow \operatorname{End}_{\mathbb{Q}} \mathcal{P}(\mathcal{Y}_t)$. However, the full endomorphism algebra of an abelian surface is never an imaginary quadratic field (see [BL04, Ex. 9.10(4)], for example) and one can check that the analytic and rational representations $A_{\mathbf{i}}$ and $R_{\mathbf{i}}$ defined below yield an element of $\operatorname{End}_{\mathbb{Q}} \mathcal{P}(\mathcal{Y}_t)$. It is then easy to see that the endomorphism algebra of the general member of our family agrees with the (indefinite) quaternion algebra F.

Abelian varieties with given endomorphism structure have been intensely studied, notably by Shimura [Shi63]. Shimura explicitly constructs moduli spaces for such families in much greater generality than we require here. However, his results specialise to our situation. To emulate his construction, we begin by observing that since $F \otimes \mathbb{R} \cong M_2(\mathbb{R})$, we can see F as a subalgebra of $M_2(\mathbb{R})$. The following matrices show the relation between the embedding $F \hookrightarrow M_2(\mathbb{R})$, the analytic representation $F \hookrightarrow M_2(\mathbb{C})$ and the rational representation $F \hookrightarrow M_4(\mathbb{Q})$

$$\mathbf{i} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix}, \quad A_{\mathbf{i}} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \qquad R_{\mathbf{i}} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & -2 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix};$$

$$\mathbf{j} = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}, \quad A_{\mathbf{j}} = \begin{pmatrix} -i\sqrt{3} & 0 \\ 0 & i\sqrt{3} \end{pmatrix}, \quad R_{\mathbf{j}} = \begin{pmatrix} -1 & 2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 2 & 1 \end{pmatrix};$$

$$\mathbf{ij} = \begin{pmatrix} 0 & \sqrt{6} \\ \sqrt{6} & 0 \end{pmatrix}, \quad A_{\mathbf{ij}} = \begin{pmatrix} 0 & i\sqrt{3} \\ -2i\sqrt{3} & 0 \end{pmatrix}, \quad R_{\mathbf{ij}} = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 \\ 2 & 2 & 0 & 0 \\ -4 & 2 & 0 & 0 \end{pmatrix}.$$

By checking which elements of F have integral rational representation, one can see that the endomorphism ring $\mathrm{End}_{\mathbb{Q}}\,\mathcal{P}(\mathcal{Y}_t)$ of the general member of our family agrees with the maximal order $\mathfrak{M}$ defined above.

Proceeding as in the case of the Wollmilchsau family and writing $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{ij}$ for an element of F, we note that, by the Skolem-Noether theorem, the quaternionic and the Rosati involution are conjugate. It is not difficult to check that, here, the Rosati involuation is given by

$$x' := \mathbf{j}^{-1}\overline{x}\mathbf{j} = x_0 + x_1\mathbf{i} - x_2\mathbf{j} + x_3\mathbf{ij}, \tag{7}$$

where $\overline{x} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{ij}$ is the usual conjugation in F. Note that the Rosati involution in $F \hookrightarrow M_2(\mathbb{R})$ agrees with transposition and that, under the embedding $F \hookrightarrow M_2(\mathbb{C})$ given by the analytic representation, it is again the restriction of the involution

$$\begin{array}{ccc} M_2(\mathbb{C}) & \to & M_2(\mathbb{C}) \\ B & \mapsto & A^{-1}B^H A \end{array}, \quad \text{for } A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{8}$$

Proof of Proposition 4.6. Let us construct the Shimura family $\mathbb{H}/\Delta(2,6,6)$. Following [Shi63], one can define the isomorphism

$$\begin{array}{rccl} \Phi: & \mathfrak{M} & \longrightarrow & \Lambda_t \\ & a & \longmapsto & A_a \cdot y, \qquad y = \left(\begin{smallmatrix} 2f \\ 2 \end{smallmatrix}\right) \end{array}$$

where A_a denotes the analytic representation of a, and check that $E(\Phi(a), y) = \mathrm{tr}(a \cdot T)$ for $T = \frac{1}{3}\mathbf{j} \in F$. The family of abelian varieties A with a $(1,2)$ polarization together with an embedding $\mathfrak{M} \hookrightarrow \mathrm{End}\,A$ and Rosati involution induced by (7) is then given by the Shimura curve $\mathbb{H}/\Gamma(T,\mathfrak{M})$, where $\Gamma(T,\mathfrak{M})$ agrees with the group of elements of norm 1 of $\mathfrak{M}$. By [Tak77] this is a quadrilateral group of signature $\langle 0; 2, 2, 3, 3\rangle$.

However, for each such variety A, there exist two different embeddings $F \hookrightarrow \mathrm{End}_{\mathbb{Q}}A$ which differ by quaternion conjugation on F. As a consequence the map $\mathbb{H}/\widetilde{\Gamma}(T,\mathfrak{M}) \to \mathcal{A}_{2,(1,2)}$ has degree 2, and the Shimura curve constructed above is a double cover of its image, which uniformised by the triangle group $\Delta(2,6,6)$ extending $\Gamma(T,\mathfrak{M})$ (see [Tak77]).

Now, the Prym-Torelli image of $\overline{\mathcal{Y}}$ lies entirely in this family and the proposition follows. □

Remark 4.7. *Cyclic coverings of this type are well-known and have been intensely studied. For example, it immediately follows from the results of Deligne and Mostow [DM86, §14.3] that the C_6-family $\mathcal{Y}$ is parametrised by $\mathbb{H}/\Delta(2,6,6)$. More precisely, the monodromy data of the C_6-family yields (using their notation) $\mu_1 = \mu_2 = {}^1\!/_3$, $\mu_3 = {}^1\!/_2$, and $\mu_4 = {}^5\!/_6$, hence we obtain a map from $\mathbb{P}^*$ into $\mathbb{H}/\Delta(3,6,6)$. Taking the quotient by the additional symmetry in the branching data here present, it descends to a map from the basis of $\mathcal{Y}$ into $\mathbb{H}/\Delta(2,6,6)$, as above.*

Recall that a (compact hyperbolic) triangle group is a Fuchsian group constructed in the following way. Let l, m and n be positive integers such that $1/l + 1/m + 1/n < 1$ and consider a hyperbolic triangle T in the hyperbolic plane with vertices v_l, v_m and v_n with angles π/l, π/m and π/n respectively. The subgroup $\Delta(l,m,n)$ of $\mathrm{PSL}_2(\mathbb{R})$ generated by the positive rotations through angles $2\pi/l$, $2\pi/m$ and $2\pi/n$ around v_l, v_m and v_n respectively is called a *triangle group of signature* (l,m,n). The triangle T is unique up to conjugation in $\mathrm{PSL}_2(\mathbb{R})$ and, therefore, so is the associated triangle group described above ([Bea83, §7.12]). Note that the quadrilateral consisting of the union of T and any of its reflections serves as a fundamental domain for $\Delta(l,m,n)$ (see Figure 9 for a fundamental domain of $\Delta(2,6,6)$ inside the hyperbolic disc $\mathbb{D}$).

In our case, the period map $f = f(t)$ from $\overline{\mathcal{Y}}_t$ to the disc of radius $1/\sqrt{2}$ gives us a particular model of the Shimura curve introduced above as the quotient of this disc with the hyperbolic metric by the action of a specific triangle group $\Delta(2,6,6)$. In order to find a fundamental domain for this group, we will study the value of the period map at the special points of the compactification $\overline{\mathcal{Y}}$ of the C_6-family $\mathcal{Y}$, namely the curves $\mathcal{Y}_{1/2}$, $\overline{\mathcal{Y}}_1$ and $\overline{\mathcal{Y}}_\infty$. In particular, we will prove the following.

Proposition 4.8. *The $\Delta(2,6,6)$ group uniformising $\overline{\mathcal{Y}}$ is generated by the hyperbolic triangle with vertices $f_{1/2} = \frac{3-\sqrt{3}+i(\sqrt{3}-1)}{4}$ of angle $\pi/2$, $f_1 = \frac{1}{2}\zeta_6$ of angle $\pi/6$ and $f_\infty = 0$ of angle $\pi/6$ inside the disc of radius $1/\sqrt{2}$ (see Figure 9). They correspond to the curves $\mathcal{Y}_{1/2}$, $\overline{\mathcal{Y}}_1$ and $\overline{\mathcal{Y}}_\infty$ respectively.*

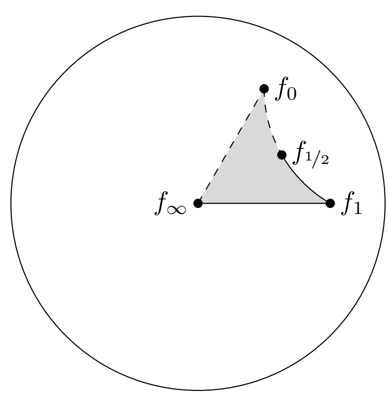

FIGURE 9. Fundamental domain of $\Delta(2,6,6)$ on the disc of radius $1/\sqrt{2}$ with vertices 0, $\frac{1}{2}$ and $\frac{1}{4}(3-\sqrt{3}) + \frac{i}{4}(\sqrt{3}-1)$ corresponding to special fibres of $\overline{\mathcal{Y}}$.

Proof. It follows from Lemma 3.2(2) that the curve $\mathcal{Y}_{1/2}$ corresponds to the point of order 2 and, therefore, $\overline{\mathcal{Y}}_1$ and $\overline{\mathcal{Y}}_\infty$ correspond to the two points of order 6. Consider (5) giving the period matrix $\Pi_t^{\mathcal{Y}}$ of the general member of the C_6-family.

Using Lemma 3.8 and the fact that $A_{\beta\mathcal{Y}}\mathcal{P}\Pi_{1/2}^{\mathcal{Y}} = \mathcal{P}\Pi_{1/2}^{\mathcal{Y}} R_{\beta\mathcal{Y}}$ one gets

$$\Pi_{1/2}^{\mathcal{Y}} = \begin{pmatrix} \vartheta & \zeta_6^{-1}\vartheta & -\vartheta & \zeta_6^2\vartheta & 1 & \zeta_6^{-1} \\ 1 & \zeta_6 & -1 & \zeta_6^{-2} & 2\vartheta & 2\zeta_6\vartheta \\ 1 & \zeta_6^{-2} & 1 & \zeta_6^{-2} & 0 & 0 \end{pmatrix},$$

where

$$\vartheta = \frac{3-\sqrt{3}+\mathrm{i}(1-\sqrt{3})}{4}.$$

In the case of $\overline{\mathcal{Y}}_\infty$, it follows from Lemma 3.10 that $\int_{F^{\mathcal{Y}}} \omega_1^\infty = \int_{G^{\mathcal{Y}}} \omega_2^\infty = \int_{G^{\mathcal{Y}}} \omega_3^\infty = 0$ and one has the following period matrix

$$\Pi_\infty^{\mathcal{Y}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & \zeta_6^{-1} \\ 1 & \zeta_6 & -1 & \zeta_6^{-2} & 0 & 0 \\ 1 & \zeta_6^{-2} & 1 & \zeta_6^{-2} & 0 & 0 \end{pmatrix}.$$

In particular $f_\infty = 0$.

In the case $\overline{\mathcal{Y}}_1$, it follows again from Lemma 3.10 that $G_2^1 = \alpha F_2^1 - F_2^1$. Comparing this with the entries of the period matrix $\Pi_t^{\mathcal{Y}}$ in (4), one finds that $2f = g_2 = \zeta_6 - 1$. Therefore $f = \frac{1}{2}\zeta_6^2$ and

$$\Pi_1^{\mathcal{Y}} = \begin{pmatrix} \frac{1}{2}\zeta_6^2 & \frac{1}{2}\zeta_6 & -\frac{1}{2}\zeta_6^2 & \frac{1}{2}\zeta_6^{-2} & 1 & \zeta_6^{-1} \\ 1 & \zeta_6 & -1 & \zeta_6^{-2} & \zeta_6^2 & -1 \\ 1 & \zeta_6^{-2} & 1 & \zeta_6^{-2} & 0 & 0 \end{pmatrix}.$$

Now, since $f_\infty = 0$ is a point of order 6 of $\Delta(2,6,6)$, the point $\frac{1}{2}\zeta_6^2$ corresponding to $\overline{\mathcal{Y}}_1$ (respectively the point $\frac{3-\sqrt{3}+\mathrm{i}(1-\sqrt{3})}{4}$ corresponding to $\mathcal{Y}_{1/2}$) is equivalent to $f_1 = \frac{1}{2}\zeta_6$ (respectively to $f_{1/2} = \frac{3-\sqrt{3}+\mathrm{i}(\sqrt{3}-1)}{4}$). □

5. Orbifold points in W_D

In this section we will finally determine the orbifold points in W_D. By Proposition 3.5, these correspond precisely to the fibres of the Wollmilchsau family $\mathcal{X}$ and of the C_6-family $\mathcal{Y}$ whose Prym variety admits proper real multiplication by $\mathcal{O}_D$, together with an eigenform for real multiplication having a 4-fold zero at a fixed point of the Prym involution. Remember that $\mathcal{O}_D$ is defined as $\mathbb{Z}[T]/(T^2+bT+c)$, where $D = b^2 - 4c$. In particular $\mathcal{O}_D$ is generated as a $\mathbb{Z}$-module by

$$T := \begin{cases} \dfrac{\sqrt{D}}{2}, & \text{if } D \equiv 0 \bmod 4\,; \\ \dfrac{1+\sqrt{D}}{2}, & \text{if } D \equiv 1 \bmod 4\,. \end{cases}$$

Let D be a discriminant with conductor f_0. The number of orbifold points on W_D of orders 2, 3, 4 and 6 are given by the following formulas.

$$h_2(D) := \begin{cases} 0\,, & D \equiv 1 \bmod 4 \quad \text{or} \quad D = 8, 12 \\ |\mathcal{H}_2(D)|/24\,, & \text{otherwise} \end{cases}$$

$$h_3(D) := \begin{cases} 0\,, & D = 12 \\ |\mathcal{H}_3(D)|\,, & \text{otherwise} \end{cases}$$

$$h_4(D) := \begin{cases} 1\,, & D = 8 \\ 0\,, & \text{otherwise} \end{cases}$$

$$h_6(D) := \begin{cases} 1\,, & D = 12 \\ 0\,, & \text{otherwise} \end{cases}$$

where

$$\mathcal{H}_2(D) := \{(a,b,c) \in \mathbb{Z}^3 \ : a^2 + b^2 + c^2 = D\ ,\ \gcd(a,b,c,f_0) = 1\,\},\ \text{and}$$

$$\begin{aligned}\mathcal{H}_3(D) := \{(a,b,c) \in \mathbb{Z}^3 \ : \ & 2a^2 - 3b^2 - c^2 = 2D\ ,\ \gcd(a,b,c,f_0) = 1\ , \\ & -3\sqrt{D} < a < -\sqrt{D}\ ,\ c < b \leq 0\ , \\ & (4a - 3b - 3c < 0) \vee (4a - 3b - 3c = 0\ \wedge\ c < 3b)\,\}.\end{aligned}$$

5.1. **Points of order 2 and 4.**

Theorem 5.1. *The curve W_8 has one orbifold point of order 4. Moreover, no other curve W_D has orbifold points of order 4.*

Let $D \neq 8, 12$ be a discriminant with conductor f_0. The number of orbifold points of order 2 in W_D is the generalised class number $h_2(D)$ defined above.

Let us recall that the Prym image of any fibre of the Wollmilchsau family $\mathcal{X}$ is given by $\mathcal{P}(\mathcal{X}_t) = \mathbb{C}^2/\Lambda$, where $\Lambda = \mathcal{P}\Pi^{\mathcal{X}} \cdot \mathbb{Z}^4$ for

$$\mathcal{P}\Pi^{\mathcal{X}} = \begin{pmatrix} \frac{-1-i}{2} & 1 & 1 & 0 \\ 1 & -1-i & 0 & 2 \end{pmatrix}$$

and that we have $\operatorname{End}_{\mathcal{Q}} \mathcal{P}(\mathcal{X}_t) \cong M_2(\mathcal{Q}[\mathrm{i}])$.

We will first study the possible embeddings of $\mathcal{O}_D$ in $\operatorname{End} \mathcal{P}(\mathcal{X}_t)$ as self-adjoint endomorphisms.

Lemma 5.2. *Let A be an element of $\operatorname{End} \mathcal{P}(\mathcal{X}_t)$. The following are equivalent:*

(i) *A is a self-adjoint endomorphism such that $A^2 = D$;*
(ii) *$A := A_{\sqrt{D}}(a,b,c) = a \cdot A_\gamma + b \cdot A_\delta + c\mathrm{i} \cdot A_{\gamma\delta}$ for some $a, b, c \in \mathbb{Z}$ such that $a^2 + b^2 + c^2 = D$.*

Proof. By Proposition 4.4, any element of $\operatorname{End}_{\mathbb{Q}} \mathcal{P}(\mathcal{X}_t)$ can be written as $A = a \cdot A_\gamma + b \cdot A_\delta + c \cdot A_{\gamma\delta} + d \cdot \mathrm{Id}$, with $a, b, c, d \in \mathbb{Q}[\mathrm{i}]$. By (3) it is clear that A is self-adjoint if and only if $a, b, d \in \mathbb{Q}$ and $c \in \mathbb{Q} \cdot \mathrm{i}$. On the other hand, only scalars or pure quaternions satisfy $A^2 \in \mathbb{Q}$, hence $d = 0$. A simple calculation shows that this implies $D = A^2 = a^2 + b^2 + c^2$.

Now, one can check that the rational representation of such an element is given by

$$R_{\sqrt{D}}(a,b,c) = \begin{pmatrix} a+b+c & -2c & 0 & 2a+2c \\ a & -a-b-c & -a-c & 0 \\ 0 & 2b & a+b+c & 2a \\ -b & 0 & -c & -a-b-c \end{pmatrix},$$

therefore A induces an endomorphism if and only if $a, b, c \in \mathbb{Z}$. □

The analytic representation

$$A_{\sqrt{D}}(a,b,c) = \begin{pmatrix} b & a \cdot \frac{1-\mathrm{i}}{2} - c \cdot \frac{1+\mathrm{i}}{2} \\ a(1+\mathrm{i}) - c(1-\mathrm{i}) & -b \end{pmatrix}$$

has eigenvectors

$$\begin{pmatrix} \frac{-1+\mathrm{i}}{2} \cdot \frac{a - c\,\mathrm{i}}{b+\sqrt{D}} \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{-1+\mathrm{i}}{2} \cdot \frac{a - c\,\mathrm{i}}{b-\sqrt{D}} \\ 1 \end{pmatrix}. \tag{9}$$

The eigenvectors (almost) determine the triple (a, b, c) and the discriminant D.

Lemma 5.3. *$A_{\sqrt{D}}(a,b,c)$ and $A_{\sqrt{D'}}(a',b',c')$ have the same eigenvectors if and only if*

- (i) *$D = m^2E$ and $D' = m'^2E$ for some discriminant E, with* $\gcd(m, m') = 1$*, and*
- (ii) *Both (a, b, c) and (a', b', c') are integral multiples of a triple $(a_0, b_0, c_0) \in \mathbb{Z}^3$ with $a_0^2 + b_0^2 + c_0^2 = D_0$.*

In particular, $A_{\sqrt{D}}(a,b,c)$ and $A_{\sqrt{D}}(a',b',c')$ have the same eigenvectors if and only if $(a,b,c) = \pm(a,b,c)$.

Proof. Suppose $A_{\sqrt{D}}(a,b,c)$ and $A_{\sqrt{D'}}(a',b',c')$ have the same eigenvectors, so that

$$\frac{a - c\,\mathrm{i}}{b+\sqrt{D}} = \frac{a' - c'\,\mathrm{i}}{b' \pm \sqrt{D'}}.$$

This immediately implies that there has to be some discriminant E such that $D = m^2E$ and $D' = m'^2E$, where we choose $\gcd(m, m') = 1$.

The equality above is equivalent to

$$ab' \pm am'\sqrt{E} = a'b + a'm\sqrt{E}$$
$$cb' \pm cm'\sqrt{E} = c'b + c'm\sqrt{E}.$$

Since E is not a square, this means $am' = \pm a'm$, $ab' = a'b$, $cm' = \pm c'm$ and $cb' = c'b$. Since m and m' are coprime we have

$$a = ma_0, \qquad b = mb_0, \qquad c = mc_0, \quad \text{and}$$
$$a' = \pm m'a_0, \qquad b' = \pm m'b_0, \qquad c' = \pm m'c_0.$$

for some triple $(a_0, b_0, c_0) \in \mathbb{Z}^3$. dividing both sides of $a^2 + b^2 + c^2 = D$ by m^2 one has $a_0^2 + b_0^2 + c_0^2 = D_0$.

The converse is immediate. □

Lemma 5.4. *Suppose $\mathcal{P}(\mathcal{X}_t)$ admits real multiplication by $\mathcal{O}_D$. Then $D \equiv 0 \bmod 4$.*

Moreover, there is a bijection between the choices of real multiplication $\mathcal{O}_D \hookrightarrow \operatorname{End}\mathcal{P}(\mathcal{X}_t)$ and the choices of triples (a, b, c) as in Lemma 5.2.

Proof. Let $\mathcal{O}_D \hookrightarrow \operatorname{End}\mathcal{P}(\mathcal{X}_t)$ be a choice of real multiplication. The rational representation of the element $T \in \mathcal{O}_D$ will be given by $R_{\sqrt{D}}(a,b,c)/2$ or $(\mathrm{Id} + R_{\sqrt{D}}(a,b,c))/2$ for some (a,b,c) satisfying the conditions of Lemma 5.2, depending on whether $D \equiv 0$ or $1 \bmod 4$ respectively. Therefore

$$R_D(a,b,c) = \begin{cases} \begin{pmatrix} \frac{a+b+c}{2} & -c & 0 & a+c \\ \frac{a}{2} & \frac{-a-b-c}{2} & \frac{-a-c}{2} & 0 \\ 0 & b & \frac{a+b+c}{2} & a \\ \frac{-b}{2} & 0 & \frac{-c}{2} & \frac{-a-b-c}{2} \end{pmatrix}, & \text{if } D \equiv 0 \bmod 4, \\ \begin{pmatrix} \frac{1+a+b+c}{2} & -c & 0 & a+c \\ \frac{a}{2} & \frac{1-a-b-c}{2} & \frac{-a-c}{2} & 0 \\ 0 & b & \frac{1+a+b+c}{2} & a \\ \frac{-b}{2} & 0 & \frac{-c}{2} & \frac{1-a-b-c}{2} \end{pmatrix}, & \text{if } D \equiv 1 \bmod 4. \end{cases}$$

A simple parity check shows that $R_D(a,b,c)$ is always integral for $D \equiv 0 \bmod 4$ and never integral for $D \equiv 1 \bmod 4$.

Conversely, every choice of (a,b,c) gives a different embedding $\mathcal{O}_D \hookrightarrow \operatorname{End}\mathcal{P}(\mathcal{X}_t)$ by Lemma 5.3. □

Lemma 5.5. *Let $D \equiv 0 \bmod 4$ be a discriminant with conductor f_0. A form ω is an eigenform for real multiplication by $\mathcal{O}_D$ if and only if it is the eigenform of some $A_{\sqrt{D}}(a,b,c)$ with $\gcd(a,b,c,f_0) = 1$.*

Proof. By the previous lemma, any choice of real multiplication corresponds to a triple $(a,b,c) \in \mathbb{Z}^3$ as in Lemma 5.2.

By Lemma 5.3, such an embedding $\mathcal{O}_D \hookrightarrow \operatorname{End}\mathcal{P}(\mathcal{X}_t)$, $T \mapsto A_D(a,b,c) := A_{\sqrt{D}}(a,b,c)/2$ is proper if and only if $\gcd(a,b,c,f_0) = 1$. □

Proof of Theorem 5.1. By Lemma 5.5, the set $\mathcal{H}_2(D)$ counts choices of proper real multiplication $\mathcal{O}_D \hookrightarrow \operatorname{End}\mathcal{P}(\mathcal{X}_t)$. Since, by Lemma 5.3, $\pm(a,b,c)$ give the same eigenforms, there are exactly $|\mathcal{H}_2(D)|$ eigenforms for real multiplication in $\mathcal{P}(\mathcal{X}_t)$ for each $D \equiv 0 \bmod 4$, up to scaling. By [Möl14, Prop. 4.6], each of them corresponds precisely to one element in some $\mathbb{P}\Omega(\mathcal{X}_t)^-(4)$. Recall also that, for each $t \in \mathbb{P}^*$, the isomorphism induced by the matrix Q_t, defined in (2), allows us to see the four differentials of $\mathcal{X}_t$ given by Lemma 3.4 in the basis of differentials associated to $\mathcal{P}\Pi^{\mathcal{X}}$.

In the case $D = 8$, one has $|\mathcal{H}_2(8)| = |\{(\pm 2, \pm 2, 0), (\pm 2, 0, \pm 2), (0, \pm 2, \pm 2)\}| = 12$. Using Q_t, it is easy to see that the eigenforms associated to the elements of $\mathcal{H}_2(8)$ correspond to the elements of $\mathbb{P}\Omega(\mathcal{X}_{-1})^-(4)$. More precisely, these eigenforms coincide, up to scaling, with the images $Q_t(\omega_1^{\mathcal{X}})$, $Q_t(\omega_2^{\mathcal{X}})$, $Q_t(-\omega_1^{\mathcal{X}} + \omega_2^{\mathcal{X}})$ and $Q_t(-t\omega_1^{\mathcal{X}} + \omega_2^{\mathcal{X}})$, for $t = -1, 1/2, 2$ (recall that $\mathcal{X}_2 \cong \mathcal{X}_{-1} \cong \mathcal{X}_{1/2}$). For example, by (9) the matrix $A_{\sqrt{8}}(2,2,0)$ has as an eigenvector $(\frac{1-i}{-2-\sqrt{8}}, 1)^\top$, which is a multiple of $Q_2(\omega_1^{\mathcal{X}})$. As a consequence of Lemma 3.7, the curve W_8 has one orbifold point of order 4 and no orbifold points of order 2. In particular, no other W_D can contain a point of order 4.

Arguing the same way for $D = 12$ and using Lemma 3.8, one finds the (unique) orbifold point of order 6 on W_{12} in accordance with Theorem 5.6.

Now, let $D \neq 8, 12$. By Proposition 3.5, we know that $\mathcal{X}_t \ncong \mathcal{X}_{-1}$. As, by Lemma 3.4, for each $t \in \mathbb{P}^*$ the set $\mathbb{P}\Omega(\mathcal{X}_t)^-(4)$ has four elements, by Lemma 3.2, we have to divide $|\mathcal{H}_2(D)|$ by $4 \cdot 6 = 24$ to get the correct number of orbifold points. $\square$

5.2. **Points of order 3 and 6.**

Theorem 5.6. *The curve W_{12} has one orbifold point of order 6. Moreover, no other curve W_D has orbifold points of order 6.*

Let $D \neq 12$ be a discriminant with conductor f_0. The number of orbifold points of order 3 in W_D is the generalised class number $h_3(D)$ defined above.

In the case of the C_6-family $\mathcal{Y}$ we are, by Lemma 3.4, only interested in the case where $\omega_2^{\mathcal{Y}}$ is an eigenform for real multiplication. Using the bases constructed in Lemmas 3.3 and 3.6, we get the following.

Lemma 5.7. *The curve $\mathcal{Y}_t$ is an orbifold point of W_D if and only if the matrix*

$$A_D \coloneqq \begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix}$$

is the analytic representation of an endomorphism of $\mathcal{P}(\mathcal{Y}_t)$ and $A_{D'}$ is not for all discriminants D' dividing D.

The orbifold order of $\mathcal{Y}_t$ is 6 if $\mathcal{Y}_t \cong \mathcal{Y}_{1/2}$ and 3 otherwise.

Proof. The form ω_2 is an eigenform for real multiplication by $\mathcal{O}_D$ on $\mathcal{P}(\mathcal{Y}_t)$ if and only if there is a matrix $\left(\begin{smallmatrix} T & 0 \\ \gamma & -T \end{smallmatrix}\right)$ for some $\gamma \in \mathbb{C}$ representing a self-adjoint endomorphism of $\mathcal{P}(\mathcal{Y}_t)$ and, moreover, the corresponding action of $\mathcal{O}_D$ is proper. By (8), the self-adjoint condition implies $\gamma = 0$. Moreover, the action of $\mathcal{O}_D$ is proper if and only if $A_{D'}$ does not induce an endomorphism for every discriminant $D'|D$.

The claim about the orbifold order follows from Proposition 3.1 and Proposition 3.5. $\square$

Using the period matrix $\mathcal{P}(\mathcal{Y}_t)$ we can compute the rational representation R_D for such an A_D in terms of f and find conditions for R_D to be integral. Remember that the parameter f lives in the disc of radius $1/\sqrt{2}$.

Proposition 5.8. *Let $f \in \mathbb{C}$ such that $|f|^2 < 1/2$ and let $\mathcal{P}(\mathcal{Y}_t)$ be as above. The matrix A_D induces a self-adjoint endomorphism of the corresponding Prym variety if and only if there exist integers $a, b, c \in \mathbb{Z}$ such that*

(i) $2a^2 - 3b^2 - c^2 = 2D$, *and*

(ii) $f = f(a, b, c, D) \coloneqq \dfrac{\sqrt{3}b\mathrm{i} + c}{2(a - \sqrt{D})}$.

Proof. Given an element of $\operatorname{End}_{\mathbb{Q}} \mathcal{P}(\mathcal{Y}_t)$ with analytic representation A, its rational representation R is given by

$$R = \begin{pmatrix} \mathcal{P}(\mathcal{Y}_t) \\ \overline{\mathcal{P}(\mathcal{Y}_t)} \end{pmatrix}^{-1} \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} \begin{pmatrix} \mathcal{P}(\mathcal{Y}_t) \\ \overline{\mathcal{P}(\mathcal{Y}_t)} \end{pmatrix}.$$

Suppose that A_D induces a self-adjoint endomorphism. In particular, the matrix $A_{\sqrt{D}} = \left(\begin{smallmatrix} \sqrt{D} & 0 \\ 0 & -\sqrt{D} \end{smallmatrix}\right)$ also induces an endomorphism and a tedious but straightforward calculation shows that the corresponding rational representation is

$$R_{\sqrt{D}} = \begin{pmatrix} B_1 & 0 & B_3 & B_2 \\ 0 & B_1 & B_2 & B_4 \\ 2B_4 & -2B_2 & -B_1 & 0 \\ -2B_2 & 2B_3 & 0 & -B_1 \end{pmatrix},$$

where

$$\begin{aligned} B_1 &= \frac{\sqrt{D}(2|f|^2+1)}{2|f|^2-1}\,, \\ B_2 &= -\frac{2\sqrt{3}\sqrt{D}(|f|^2-f^2)\mathrm{i}}{3f(2|f|^2-1)}\,, \\ B_3 &= \frac{\sqrt{3}\sqrt{D}(|f|^2-f^2)\mathrm{i}}{3f(2|f|^2-1)} + \frac{\sqrt{D}(|f|^2+f^2)}{f(2|f|^2-1)} \quad \text{and} \\ B_4 &= \frac{\sqrt{3}\sqrt{D}(|f|^2-f^2)\mathrm{i}}{3f(2|f|^2-1)} - \frac{\sqrt{D}(|f|^2+f^2)}{f(2|f|^2-1)}. \end{aligned}$$

We define $a \coloneqq B_1 \in \mathbb{Z}$ and from the expression above we get that

$$|f|^2 = \frac{1}{2}\cdot\frac{a+\sqrt{D}}{a-\sqrt{D}}\,. \tag{10}$$

Moreover, since $|f|^2 - f^2 = -2\mathrm{i}\cdot f\,\mathrm{Im}\,f$, $|f|^2+f^2 = 2f\,\mathrm{Re}\,f$ and $2|f|^2-1 = \frac{2\sqrt{D}}{a-\sqrt{D}}$, the expressions above imply

$$b \coloneqq B_2 = \frac{2(a-\sqrt{D})\,\mathrm{Im}(f)}{\sqrt{3}} \quad \text{and} \quad c \coloneqq 2B_3 - B_2 = -2B_4 + B_2 = 2(a-\sqrt{D})\,\mathrm{Re}(f)\,,$$

so that

$$f = \frac{c+\sqrt{3}b\mathrm{i}}{2(a-\sqrt{D})}\,,$$

and (10) implies that $2a^2 - 3b^2 - c^2 = 2D$, as claimed.

Conversely, suppose that $a, b, c \in \mathbb{Z}$ satisfy the conditions of the proposition and define $f = f(a,b,c,D)$ as above. The rational representation of A_D (at the point corresponding to f) is given by $R_D = R_{\sqrt{D}}/2$ or $(\mathrm{Id} + R_{\sqrt{D}})/2$, depending on whether $D \equiv 0$ or $1 \bmod 4$, respectively, and therefore

$$R_D = \begin{cases} \begin{pmatrix} \frac{a}{2} & 0 & \frac{b+c}{2} & b \\ 0 & \frac{a}{2} & b & \frac{b-c}{2} \\ b-c & -2b & -\frac{a}{2} & 0 \\ -2b & b+c & 0 & -\frac{a}{2} \end{pmatrix}, & \text{if } D \equiv 0 \bmod 4, \\[2em] \begin{pmatrix} \frac{1+a}{2} & 0 & \frac{b+c}{2} & b \\ 0 & \frac{1+a}{2} & b & \frac{b-c}{2} \\ b-c & -2b & \frac{1-a}{2} & 0 \\ -2b & b+c & 0 & \frac{1-a}{2} \end{pmatrix}, & \text{if } D \equiv 1 \bmod 4. \end{cases}$$

Considering the equality $2a^2 - 3b^2 - c^2 \equiv 2D \bmod 8$, one sees that

- a, b and c are even if $D \equiv 0 \bmod 4$, and
- a is odd and b and c are even if $D \equiv 1 \bmod 4$

and therefore $R_D \in M_4(\mathbb{Z})$ in both cases. □

For each discriminant D one can count how many points $f(a,b,c,D)$ in the fundamental domain of $\Delta(2,6,6)$ satisfy the previous conditions. Recall from Section 4.2 that we consider the fundamental domain for the triangle group $\Delta(2,6,6)$ depicted in Figure 9.

Lemma 5.9. *Let $\widetilde{\mathcal{H}}_3(D)$ be the set of triples of integers (a,b,c) such that*

(i) $2a^2 - 3b^2 - c^2 = 2D$;
(ii) $-3\sqrt{D} < a < -\sqrt{D}$;
(iii) $c < b \leq 0$;
(iv) *Either* $4a - 3b - 3c < 0$, *or* $4a - 3b - 3c = 0$ *and* $c < 3b$.

The set $\widetilde{\mathcal{H}}_3(D)$ agrees with the triples (a,b,c) in Proposition 5.8 that yield a point $f(a,b,c,D)$ in the fundamental domain of $\Delta(2,6,6)$.

Remark 5.10. *Note that $\widetilde{\mathcal{H}}_3(D)$ agrees with the set $\mathcal{H}_3(D)$ defined above except for the condition on the* gcd*. This condition will ensure that the embedding of $\mathcal{O}_D$ into* $\operatorname{End}\mathcal{P}(\mathcal{Y}_t)$ *is proper.*

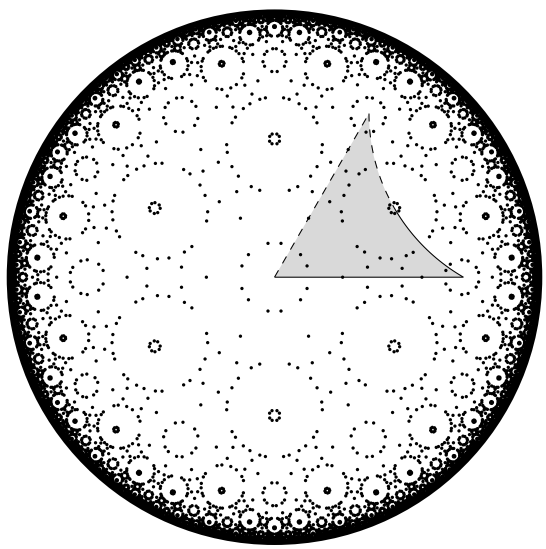

FIGURE 10. Points in the disc of radius $1/\sqrt{2}$ satisfying the conditions of Proposition 5.8 for $D = 3257$ together with the fundamental domain of $\Delta(2,6,6)$.

Proof. Recall that we are using the fundamental domain depicted in Figure 9, whose vertices have been calculated in Proposition 4.8. Condition *(ii)* ensures that $0 \leq |f|^2 \leq 1/4$ and condition *(iii)* that $0 \leq \arg f < \frac{\pi}{3}$. Now, the geodesic joining f_0 and f_1 is an arc of circumference $|z - (3+\sqrt{3}\mathrm{i})/4|^2 = 1/4$. Therefore, f lives on the (open) half-disc containing the origin, determined by this geodesic, if and only if

$$\left| f - \frac{3+\sqrt{3}\mathrm{i}}{4} \right|^2 = \left(\frac{c}{2(a-\sqrt{D})} - \frac{3}{4} \right)^2 + \left(\frac{\sqrt{3}b}{2(a-\sqrt{D})} - \frac{\sqrt{3}}{4} \right)^2 \geq \frac{1}{4}.$$

Expanding this expression and using the previous conditions, one gets the first part of condition *(iv)*. Since the sides joining f_1 and $f_{1/2}$, and $f_{1/2}$ and f_0 are identified by an element of order 2 in $\Delta(2,6,6)$, we need to count only the points f that lie in one of them, say the arc of the geodesic joining f_1 and $f_{1/2}$. Proceeding as before we obtain the second part of condition *(iv)*. □

Proof of Theorem 5.6. First note that if $D = g^2 D'$, then

(11) $\quad f(a,b,c,D) = f(a',b',c',D') \quad$ if and only if $a = ga',\ b = gb'$ and $c = gc'$.

Since 12 is a fundamental discriminant, Lemma 5.7 and Lemma 5.9 imply that W_{12} has one orbifold point of order 6. Moreover, this is the only curve with an orbifold point of order 6 because, by (11) above, the point $f(a,b,c,D)$ can only correspond to $t = 1/2$ if one has $D = f_0^2 D_0$ for $D_0 = 12$.

Now let $D \neq 12$. By Lemma 5.7 and Lemma 5.9, we only need to prove that $\mathcal{H}_3(D)$ is the set of triples in $\widetilde{\mathcal{H}}$ which are not contained in any $\widetilde{\mathcal{H}}_3(D')$, for discriminants $D'|D$. This is true since, by (11), $(a,b,c) \in \widetilde{\mathcal{H}}_3(D)$ is not contained in any $\widetilde{\mathcal{H}}_3(D')$ if and only if $\gcd(a,b,c,f_0) = 1$. □

6. Examples

Example 1 (W_{12} and W_{20}). In [Möl14, Ex. 4.4] it is shown that W_{12} has genus zero, two cusps and one orbifold point of order 6, and that W_{20} has genus zero, four cusps and one elliptic point of order 2. Our results agree with his. These are the curves $V(S_1)$ and $V(S_2)$ in [McM06].

Example 2 (W_8). By Theorem 5.1 and Theorem 5.6, we find that W_8 has one orbifold point of order 3 and one orbifold point of order 4. By [LN14, Thm. C.1] the number of cusps is $C(W_8) = 1$ and by [Möl14, Thm. 0.2] the Euler characteristic is $\chi(W_8) = -5/12$. We can then use (1) to compute its genus as $g(W_8) = 0$.

Example 3 (W_{2828}). Theorem 5.1 and Theorem 5.6 also tell us that W_{2828} has six orbifold points of order 2. They correspond to the $|\mathcal{H}_2(2828)| = 144$ eigenforms for real multiplication by $\mathcal{O}_{2828}$ in $\mathcal{P}(\mathcal{X}_t)$, as in (9), divided by 24. In Figure 11, we depict the first coordinate of the elements of $\mathcal{H}_2(2828)$ in the complex plane together with the unit circle.

As for the orbifold points of order 3, there are twenty of them. They correspond to the twenty points in the Shimura curve $\mathbb{D}/\Delta(2,6,6)$ admitting proper real multiplication by $\mathcal{O}_{2828}$. In Figure 11, we depict the preimage of these 20 points in $\mathbb{D}$, that is the points $f(a,b,c,2828)$ as in Proposition 5.8.

The number of cusps is $C(W_{2828}) = 68$ and the Euler characteristic is $\chi(W_{2828}) = -8245/3$. Therefore, by (1), the genus is $g(W_{2828}) = 1333$.

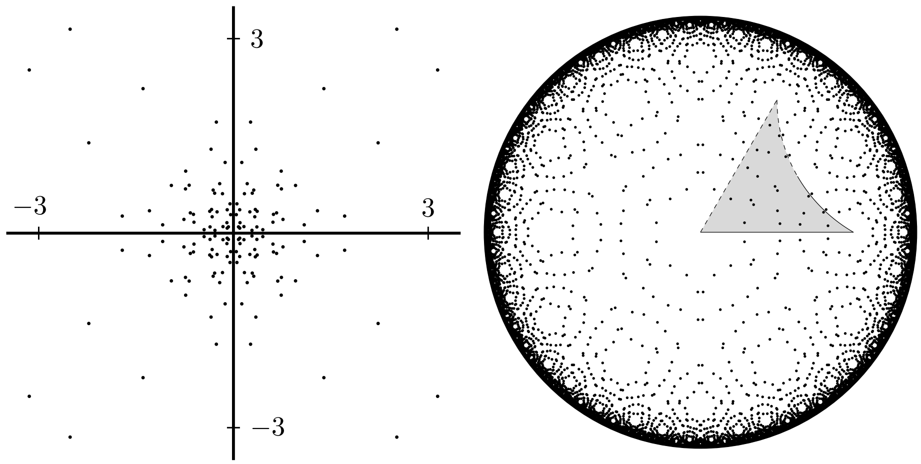

FIGURE 11. Calculation of the orbifold points of order 2 and 3 in W_{2828}.

D	χ	C	g	h_2	h_3
17	-5/3	3	0	0	1
20	-5/2	4	0	1	0
24	-5/2	4	0	1	0
28	-10/3	4	0	0	2
32	-5	7	0	0	0
33	-5	7	0	0	0
40	-35/6	6	0	1	2
41	-20/3	8	0	0	1
44	-35/6	6	0	1	2
48	-10	10	1	0	0
52	-25/2	12	1	1	0
56	-25/3	6	1	2	2
57	-35/3	11	1	0	1
60	-10	8	2	0	0
65	-40/3	12	1	0	2
68	-15	14	1	2	0
72	-25/2	10	2	1	0
73	-55/3	17	1	0	2
76	-95/6	14	1	1	2
80	-20	16	3	0	0
84	-25	16	5	2	0
88	-115/6	16	1	1	4
89	-65/3	15	4	0	1
92	-50/3	8	4	0	4
96	-30	20	6	0	0
97	-85/3	21	4	0	2
104	-125/6	10	5	3	2
105	-30	18	7	0	0
108	-45/2	14	5	1	0
112	-40	24	9	0	0
113	-30	18	6	0	3
116	-75/2	20	9	3	0
120	-85/3	12	8	2	2
124	-100/3	16	9	0	2
128	-40	22	10	0	0
129	-125/3	25	9	0	1
132	-45	30	8	2	0
136	-115/3	20	9	2	2
137	-40	22	9	0	3
140	-95/3	12	9	2	4
145	-160/3	32	11	0	2
148	-125/2	36	14	1	0
152	-205/6	12	10	3	4
153	-50	30	11	0	0
156	-130/3	16	14	0	2
160	-70	42	15	0	0
161	-160/3	22	16	0	2
164	-60	32	14	4	0
168	-45	16	15	2	0
172	-105/2	22	14	1	6
176	-70	30	21	0	0
177	-65	31	18	0	0
180	-75	32	22	2	0
184	-185/3	22	19	2	4
185	-190/3	26	19	0	2
188	-140/3	12	17	0	4
192	-80	36	23	0	0
193	-245/3	39	21	0	4
200	-325/6	18	17	3	4
201	-245/3	37	23	0	1
204	-65	28	19	2	0
208	-100	48	27	0	0
209	-235/3	35	22	0	2
212	-175/2	28	30	3	0
216	-135/2	32	18	3	0
217	-290/3	42	27	0	4
220	-230/3	32	22	0	4
224	-100	34	34	0	0
228	-105	46	30	2	0
232	-165/2	30	25	1	6
233	-265/3	29	29	0	5
236	-425/6	26	22	3	2
240	-120	40	41	0	0
241	-355/3	49	35	0	2
244	-275/2	52	43	3	0
248	-70	14	26	4	6
249	-115	45	36	0	0
252	-80	24	29	0	0
257	-100	34	33	0	3
260	-120	48	36	4	0
264	-280/3	32	30	4	2
265	-400/3	56	39	0	2
268	-205/2	30	35	1	6
272	-120	44	39	0	0
273	-370/3	38	43	0	2
276	-150	40	55	4	0
280	-335/3	36	37	2	4
281	-125	45	40	0	3
284	-290/3	20	38	0	4
288	-150	54	49	0	0
292	-165	74	46	2	0
296	-205/2	22	38	5	6
297	-135	49	44	0	0
300	-325/3	28	40	2	2

Table 2. Topological invariants of the Prym-Teichmüller curves W_D for D up to 300.

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