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THETA DERIVATIVES AND TEICHMÜLLER CURVES (ARBEITSTAGUNG 2011)

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1. Special curves on Hilbert modular surfaces

Consider the Hilbert modular surfaces $X_D = \mathbb{H}^2/\mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee)$ where $\mathfrak{o}$ is the order of discriminant D in $K = \mathbb{Q}(\sqrt{D})$. Clearly the most special algebraic curve in X_D is the diagonal, the image of the composition $z \mapsto (z, z)$ and the projection $\pi : \mathbb{H}^2 \to X_D$. For any matrix $M \in \mathrm{GL}_2^+(K)$ one can consider the twisted diagonal $z \mapsto (Mz, M^\sigma z)$, where σ is the generator of the Galois group. The π-images of these twisted diagonals are still algebraic curves, called special curves, Shimura curves, modular curves or Hirzebruch-Zagier cycles and the literature on them is even longer than the number of names. Note that for these curves both components of the universal covering map are given by Mobius transformations.

An algebraic curve $C \to X_D$ in a Hilbert modular surface is still quite special if one asks just that (at least) *one* of the components of the universal covering map $\mathbb{H} \to \mathbb{H}^2$ should be a Mobius transformation. Equivalently, one may ask that $C \to X_D$ is totally geodesic for the Kobayashi metric and we thus call these curves *Kobayashi geodesics*. Yet equivalently, we can characterize these curves as being everywhere transversal to (at least) one of the two foliations of $\mathbb{H}^2$. See [MV10] for more equivalent conditions.

We will provide examples of these curves soon. We give one number theoretic reason why one might be interested in these curves. Consider the differential equation

$$
\begin{aligned}
L(y,t) &= \big(A(t)\, y'(t)\big)' \;+\; B(t)\, y(t) \;=\; 0\\
A(t) &= t\,(t-1)\,(t-\ell)(t-\ell^{-1}) \;=\; t^4 - \beta t^3 + \beta t^2 - t\,,\\
B(t) &= \frac{3}{4}\big(3t^2 - (\beta+\gamma)\,t + \gamma\big)
\end{aligned}
\tag{1}
$$

where

$$
\ell = \frac{31-7\sqrt{17}}{2}, \quad \beta = \ell + \ell^{-1} + 1 = \frac{1087-217\sqrt{17}}{64}, \quad \gamma = \frac{27-5\sqrt{17}}{4}\,.
$$

There is a well-known recursive procedure for finding a solution $y = \sum_{n\geq 0} a_n t^n$ of such a differential equation that involves dividing by $(n+1)^2$ when computing the n-th term. But the solution of this particular differential equation

$$
y = 1 + \frac{81-15\sqrt{17}}{16}t + \frac{4845-1155\sqrt{17}}{64}t^2 + \frac{3200225-775495\sqrt{17}}{2048}t^3 + \ldots \tag{2}
$$

has coefficients in the ring of integers $\mathfrak{o}_{\sqrt{17}}[1/2]$ ([BM10]). The differential equation is the Picard-Fuchs equation for the curve W_{17} introduced below.

2. Theta derivatives

Let $\Theta_{(m,m')}(v,Z)$ be the usual Siegel theta function on $\mathbb{C}^2 \times \mathbb{H}_2$ with characteristic $(m,m') \in (\frac{1}{2}\mathbb{Z}^2/\mathbb{Z}^2)^2$. A choice of a basis for $\mathfrak{o}_D$ determines a 'Siegel' modular embedding, i.e. map $\psi : \mathbb{H}^2 \to \mathbb{H}_2$ and equivariant with respect to an adapted group homomorphism $\Psi : \mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee) \to \mathrm{Sp}_4(\mathbb{Z})$.
In Siegel upper half space there are no distinguished directions and consequently none of the partial derivatives of Θ with respect to ε_i is distinguished. Altogether the form a vector-valued modular form. But $\mathbb{H}^2$ has two distinguished foliations and thus the restriction of $\Theta(z_1,z_2)$ to the universal covering of X_D has two distinguished partial derivatives. We denote second of these derivatives by $D_2\Theta(z_1,z_2)$. This is a modular form of weight $(1/2, 3/2)$ for some subgroup of $\mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee)$.

Theorem 2.1 ([MZ11]). *The function*

$$D_2\Theta(z_1,z_2) = \prod_{(m,m')\ \mathrm{odd}} D_2\Theta_{(m,m')}(0,\psi(z_1,z_2))$$

is a modular form for the full Hilbert modular group $\mathrm{SL}(\mathfrak{o}_D \oplus \mathfrak{o}_D^\vee)$ *of weight* $(3,9)$. *Its vanishing locus*

$$W_D = \{D_2\Theta(z_1,z_2) = 0\} \subset X_D$$

is a Kobayashi geodesic.

Sketch of proof. Being transversal to the second of the two foliations means that the derivative in the z_2-direction never vanishes. Using the heat equation this means that the third partial derivative of the theta function never vanishes on W_D (in the interior of X_D). This third derivative is a 'modular form' on W_D. The number of zeros on a compactification of W_D can thus be computed. It suffices thus to list the number of cusps of W_D and show that the vanishing orders of the third derivative at these points add up to the required number. □

3. Connection to Teichmüller curves

Teichmüller curves are algebraic curves in the moduli space of curves $\mathcal{M}_g$ that are totally geodesic for the Kobayashi (equivalently: Teichmüller) metric. In [McM03] McMullen found an interesting series of such curves W_D^{Eig} using eigenforms for real multiplication, see [McM05] for a complete classification. Precisely,

$$\begin{aligned} W_D^{\mathrm{Eig}} = \{&[X] \in \mathcal{M}_2 \,:\, \mathrm{Jac}(X)\,\text{has RM by}\,\mathfrak{o}_D, \\ &\text{a RM-eigenform}\,\omega \in H^0(X,\Omega_X^1)\,\text{has a double zero}\} \end{aligned}$$

Theorem 3.1 ([MZ11]). *These two series of curves coincide, i.e.* $W_D = W_D^{\mathrm{Eig}}$ *when considered in* $\mathcal{A}_2$.

The proof relies on two facts. First, a genus two curve equals the theta divisor in its Jacobian. Second an eigenform has a double zero if and only if the derivative of the theta function in a 'foliation' direction vanishes at a Weierstraß point.
By construction W_D^{Eig} is in $\mathcal{M}_2$, hence disjoint from the locus $P_D \subset X_D$ of reducible abelian surfaces. There are two more proofs of this fact using theta functions only.

Theorem 3.2 ([MZ11]). *The loci* W_D *and* P_D *are disjoint.*

Sketch of proof. The reducible locus is the vanishing locus of the product of all even theta functions. Its restriction to X_D is the vanishing locus of a modular form of weight $(5, 5)$. As in the proof of Theorem 2.1 one can thus calculate the number of intersection point of W_D and P_D by intersection theory. Again, a local calculation at the cusps of W_D shows that the intersection points are all located there.
For the second proof, one shows that on the reducible locus the derivatives of theta functions factor as a product of two unary theta series. They are known to vanish only at the cusps. □

Since a Kobayashi geodesic C in X_D is a Teichmüller curve if and only if C is disjoint from P_D, this provides a proof of the property Teichmüller curve using theta functions only.
Disconnecting from the world of Teichmüller curves. Given the univeral covering map $z \mapsto (z, \varphi(z))$ of a Kobayashi geodesic one can obtain more Kobayashi geodesics by twisting, i.e. considering the π-images of $z \mapsto (Mz, M^\sigma \varphi(z))$. For these curves one can ask the same questions as for the Hirzebruch-Zagier curves. Some answers are provided in the forthcoming Ph.D. thesis of C. Weiß. But this is still surely not yet the end the story.
If C is Kobayashi geodesic and $\mathcal{L}_i$ are the classes of the two foliations of X_D, then the quantity $\lambda_2 = (C \cdot \mathcal{L}_1)/(C \cdot \mathcal{L}_2)$ is invariant under twisting. Beside the case $\lambda_2 = 1$ (HZ-cycles) and $\lambda_2 = 1/3$ (from W_D) C. Weiss also showed that the Prym Teichmüller curves of [McM06] give Kobayashi geodesics with $\lambda_2 = 1/7$. A construction of these curves using Θ-functions is in progress.

4. Two compactifications

A list of cusps of W_D was needed in (some of the) proof(s) sketched above. To describe them, there is a very useful compactification $\overline{X_D}^B$ defined by Bainbridge ([Ba07]) as follows. Consider the preimage of X_D in $\mathcal{M}_2$, lift to $\Omega\overline{\mathcal{M}}_2$, the total space of the relative dualizing sheaf over the Deligne-Mumford compactification, and take $\overline{X_D}^B$ to be the normalization of the closure.
On the other hand there is Hirzebruch's compactification $\overline{X_D}^H$, the minimal smooth compactification. This compactification is toroidal, that is given by a fan, a sequence of $\alpha_n \in \mathfrak{o}_D$ totally positive with $\sigma(\alpha_n)/\alpha_n$ decreasing and invariant under multiplication by squares of units in $\mathfrak{o}_D$. The toroidal structure allows to compute easily e.g. if and at which point HZ-cycles meet the boundary.
There is also a way of realizing $\overline{X_D}^B$ as a toroidal compactification. For a fractional $\mathfrak{o}_D$ ideal $\mathfrak{a}$ let $\mathfrak{a}^*[2]$ be the set of non-zero elements in $\frac{1}{2}\mathfrak{a}/\mathfrak{a}$. We let $\widetilde{\mathrm{MM}}(\mathfrak{a}, \xi)$ be the set of $\alpha \in K$ such that the quadratic form $F(x) = \mathrm{tr}(\alpha x^2)$ is positive definite and assumes its minimum on $\mathfrak{a} + \xi$ more than once (where x and $-x$ are not distinguished). We define a *multiminimizer for* ξ to be the equivalence classes

$$\mathrm{MM}(\mathfrak{a}, \xi) = \widetilde{\mathrm{MM}}(\mathfrak{a}, \xi)/\mathbb{Q}^*$$

and we let the set of *multiminimizers* be the union of $\mathrm{MM}(\mathfrak{a}, \xi)$ over all $\xi \in a^*[2]$.

Theorem 4.1 ([MZ11]). *For any $\mathfrak{a}$, the set of multiminimizers forms a fan. The associated toroidal compactification is Bainbridge's compactification $\overline{X_D}^B$ at the cusp $\mathfrak{a}$.*

This compactification can be calculated by an easy algorithm. In fact, given one multiminimizer, the subsequent ones can be constructed using the 'slow-greater one' continued fraction algorithm. Here 'slow-greater one' continued fraction algorithm means that

$$x_{n+1} = \begin{cases} x_n - 1 & \text{if} \quad x_n > 2 \\ 1/(x_n - 1) & \text{if} \quad 2 > x_n > 1 \end{cases} .$$

Note that Hirzebruch's compactification is driven by the 'fast-minus' continued fraction algorithm

$$x = p_1 - \cfrac{1}{p_2 - \cfrac{1}{\ddots}},$$

where at each step $p_i = \lceil x_i \rceil$.

References

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