Goethe-Universität Frankfurt
Institut für Mathematik
Winter term 2020/21
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Algebra
Prof. Dr. Martin Möller
M.Sc. Riccardo Zuffetti

Übungsblatt 5

Aufgabe 1 (6 Punkte)

Let L/K be a field extension together with intermediate fields E and E' such that E/K and E'/K are finite Galois extensions.

(i) Prove that $E \cdot E'$ is finite and Galois over K, and that the homomorphism

$$\varphi\colon \operatorname{Gal}(E \cdot E'/E) \longrightarrow \operatorname{Gal}(E'/E \cap E'), \qquad \sigma \longmapsto \sigma|_{E'}$$

is an isomorphism.

(ii) Prove that the homomorphism

$$\eta\colon \operatorname{Gal}(E \cdot E'/K) \longrightarrow \operatorname{Gal}(E/K) \times \operatorname{Gal}(E'/K), \qquad \sigma \longmapsto \left(\sigma|_E, \sigma|_{E'}\right)$$

is injective. Show also that if $E \cap E' = K$, then η is surjective and hence an isomorphism.

(iii) Prove that $[E \cdot E' : K] = \frac{[E:K][E':K]}{[E \cap E':K]}$.

(iv) The previous formula is in general <u>*false*</u> if E/K and E'/K are finite but non-Galois. Show that with an example.
Hint: consider two simple extensions of $\mathbb{Q}$ *generated by two different roots of* $X^3 - 2$.

Aufgabe 2 (10 Punkte)

Let $\alpha_{s,t} := s\sqrt{3} + t\sqrt{2}$, where $s, t \in \mathbb{Q}$.

(i) Prove that $\mathbb{Q}(\alpha_{1,1})/\mathbb{Q}$ is a finite Galois extension.

(ii) Compute $[\mathbb{Q}(\alpha_{1,1}) : \mathbb{Q}]$.

(iii) Compute $\operatorname{Gal}(\mathbb{Q}(\alpha_{1,1})/\mathbb{Q})$.

(iv) Compute all intermediate fields of $\mathbb{Q}(\alpha_{1,1})/\mathbb{Q}$.

(v) Compute the minimal polynomial of $\alpha_{1,1}$.

(vi) Prove that $\mathbb{Q}(\alpha_{s,t}) = \mathbb{Q}(\alpha_{1,1})$ if and only if $s \cdot t \neq 0$. Prove also that $\mathbb{Q}(\alpha_{s,t})/\mathbb{Q}$ is finite and Galois for every $s, t \in \mathbb{Q}$.

(vii) Compute the minimal polynomial of $\alpha_{s,t}$ over $\mathbb{Q}$, for every $s, t \in \mathbb{Q}$.

Hint: you can use Aufgabe 1 and Übungsblatt 2 Aufgabe 3. Recall that $\operatorname{Gal}(\mathbb{Q}(\alpha_{s,t})/\mathbb{Q})$ *permutes the roots of the minimal polynomial of* $\alpha_{s,t}$.

Please, upload your solutions on the Olat page of this course, by **14:00** on **Tuesday, 8.12.2020**.