Goethe-Universität Frankfurt
Institut für Mathematik
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Algebra
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Übungsblatt 4

Aufgabe 1 (2 Punkte)

Let L/K be a finite field extension of degree $[L : K] = n$. Assume there is an element $\alpha \in L$ together with isomorphisms $\sigma_j \colon L \longrightarrow L$, $j = 1, \ldots, n$, satisfying $\sigma_j|_K = \mathrm{id}_K$ and $\sigma_j(\alpha) \neq \sigma_\ell(\alpha)$ for $j \neq \ell$. Show that $L = K(\alpha)$.

Aufgabe 2 (5 Punkte)

Consider $L = \mathbb{Q}(\sqrt[4]{2}, i)$ as a field extending $\mathbb{Q}$.

(i) Prove that L is a splitting field of $X^4 - 2$ over $\mathbb{Q}$.

(ii) Compute $[L : \mathbb{Q}]$, and find a basis of L as $\mathbb{Q}$-vector space.

(iii) Determine the eight $\mathbb{Q}$-automorphisms of $\mathrm{Aut}_{\mathbb{Q}}(L)$.
Hint: find where each automorphism maps the roots of $X^4 - 2$ and i.

(iv) Prove that $L = \mathbb{Q}(\sqrt[4]{2} + i)$.
Hint: use Aufgabe 1.

Aufgabe 3 (5 Punkte)

Consider $f(X) \coloneqq X^6 - 7X^4 + 3X^2 + 3$ as a polynomial in $\mathbb{Q}[X]$, as well as in $\mathbb{F}_{13}[X]$. In either case, decompose f into its irreducible factors, and determine a splitting field of f over $\mathbb{Q}$, resp. $\mathbb{F}_{13}$.

Aufgabe 4 (4 Punkte)

Let L/K be a field extension in characteristic $p > 0$ and consider an element $\alpha \in L$ that is algebraic over K. Show that α is separable over K if and only if $K(\alpha) = K(\alpha^p)$.
Hint: ($\Rightarrow$) Suppose $K(\alpha) \neq K(\alpha^p)$, prove that $X^p - \alpha^p$ is irreducible over $K(\alpha^p)$.
($\Leftarrow$) Note that if $K(\alpha) = K(\alpha^p)$, then there exists $F \in K[X]$ such that α is a root of $F(X^p) - X$. Why?

Please, upload your solutions on the Olat page of this course, by **14:00** on **Tuesday, 1.12.2020**.