Goethe-Universität Frankfurt
Institut für Mathematik
Winter term 2020/21
16. November 2020

Algebra
Prof. Dr. Martin Möller
M.Sc. Riccardo Zuffetti

Übungsblatt 3

Aufgabe 1 (4 Punkte)

Prove that $X^n + 1$ is irreducible in $\mathbb{Q}[X]$ if and only if $n = 2^k$ for some non-negative integer k. Is that true also in $\mathbb{Z}[X]$?
Hint: apply Eisenstein's criterion after a substitution.

Aufgabe 2 (4 Punkte)

(i) Prove that any algebraic closure of $\mathbb{Q}(\sqrt{11})$ is isomorphic to any algebraic closure of $\mathbb{Q}(\sqrt{-7})$.

(ii) Suppose that F/K is a field extension where F is algebraically closed. Prove that the set of elements of F which are algebraic over K is an algebraically closed field.

(iii) Let n be a positive integer. Let F be a splitting field of the set of all polynomials of degree at most n over $\mathbb{Q}$. Prove that F is not an algebraic closure of $\mathbb{Q}$.

Aufgabe 3 (4 Punkte)

Prove that every algebraically closed field consists of infinitely many elements.

Aufgabe 4 (4 Punkte)

Let $\overline{K}$ be an algebraic closure of a field K. Prove that if K is countable, so is $\overline{K}$.
Hint: prove that the explicit construction of an algebraically closed extension L of K given by the proof of Satz 2.23 is countable. Then look at the proof of Korollar 2.25.

Please, upload your solutions on the Olat page of this course, by **14:00** on **Tuesday, 24.11.2020**.