Goethe-Universität Frankfurt
Institut für Mathematik
Winter term 2020/21
10. November 2020

Algebra
Prof. Dr. Martin Möller
M.Sc. Riccardo Zuffetti

Übungsblatt 2

Aufgabe 1 (3 Punkte)

Let L/K be a field extension.

(i) Let $\alpha, \beta \in L$ be algebraic over K. Prove that $\alpha + \beta$ and $\alpha\beta$ are algebraic over K.

(ii) Suppose that $[L : K] = p$ is a prime. Prove that there exists an element $\alpha \in L$ such that $L = K(\alpha)$.

(iii) Suppose that $[L : K] = 2^k$ for some $k \in \mathbb{N}$. Let $f \in K[X]$ be a polynomial of degree 3. Prove that if f has a root in L, then it has a root already in K.

Aufgabe 2 (4 Punkte)

Prove that a field extension L/K is algebraic if and only if every subring R satisfying $K \subset R \subset L$ is a field.

Aufgabe 3 (3 Punkte)

Let F/K be a finite field extension, and let E and L be intermediate fields. We denote by $E \cdot L$ the smallest subfield of F containing E and L.

(i) Let S and T be subsets of F such that $E = K(S)$ and $L = K(T)$. Prove that

$$E \cdot L = K[S \cup T] = K(S \cup T).$$

(ii) Prove that $[E \cdot L : K] \leq [E : K][L : K]$.

(iii) Provide an example where the previous inequality is strict and an example where the previous inequality is an equality.

Aufgabe 4 (6 Punkte)

(i) Let $f(X) = a_d X^d + \cdots + a_0 \in \mathbb{Z}[X]$, with $a_d \neq 0$, and let $\frac{r}{s} \in \mathbb{Q}$ be a root of f, with $\gcd(r, s) = 1$. Prove that $r | a_0$ and $s | a_d$.

(ii) Find out which of the following polynomials are irreducible in $\mathbb{Q}[X]$.

$$X^3 - 3X - 1, \qquad 2X^3 - 3X - 1,$$
$$3X^3 - 3X - 1, \qquad X^4 + 3X^3 + X^2 - 2X + 1.$$

(iii) Prove that the polynomial $X^2Y + XY^2 - X - Y + 1$ is irreducible in $\mathbb{Q}[X, Y]$.

Please, upload your solutions on the Olat page of this course, by **14:00** on **Tuesday, 17.11.2020**.