Goethe-Universität Frankfurt
Institut für Mathematik
Winter term 2020/21
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Algebra
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Übungsblatt 13

Aufgabe 1 (4 Punkte)

(i) Let p be a prime. Prove that the equation $f(X) = 0$ is solvable by radicals over $\mathbb{F}_p$, for every separable polynomial $f \in \mathbb{F}_p[X]$.

(ii) Prove that the equation

$$X^{10} + 3X^6 + X^2 - T^2 = 0$$

is solvable by radicals over the field of fractions $\mathbb{F}_5(T)$.

(iii) Provide an example of a separable polynomial f with coefficients in a field of positive characteristic, such that the equation $f(X) = 0$ is not solvable by radicals.

Hint: Recall the Artin–Schreier Theorem and the version of "solvability by radicals" for field extensions in positive characteristic; see Bosch' book, Sections 4.8 and 6.1.

Aufgabe 2 (4 Punkte)

(i) Compute the transcendence degree of the following field extensions.

$$\mathbb{Q}(\sqrt[n]{n} \,:\, n \in \mathbb{Z}_{>0})/\mathbb{Q}, \qquad \mathbb{Q}(\sqrt[n]{\pi} \,:\, n \in \mathbb{Z}_{>0})/\mathbb{Q},$$
$$\mathbb{C}/\mathbb{R}, \qquad \mathbb{F}_{p^n}(T)/\mathbb{F}_p \text{ where } n > 0.$$

(ii) Show that every transcendence basis of $\mathbb{R}/\mathbb{Q}$ is uncountable.

(iii) Let L/K be a field extension and $\mathfrak{X}$ an algebraically independent system of L/K. Show for every intermediate field K' of L/K that is algebraic over K that $\mathfrak{X}$ is algebraically independent over K'.

(iv) Let L/K be a finitely generated field extension. Show for every intermediate field L' of L/K that the extension L'/K is finitely generated.

Please, upload your solutions on the Olat page of this course, by **14:00** on **Tuesday, 16.02.2021**.