Goethe-Universität Frankfurt
Institut für Mathematik
Sommersemester 2019
19. Juni 2019

Algebraische Geometrie II
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Übungsblatt 5

Aufgabe 1 (8 Punkte)

Let C be a compact smooth Riemann surface and let E be a rank 2 vector bundle over C. We denote by $S := \mathbb{P}(E)$ the projective bundle associated to E. It is a complex compact surface equipped with a map $p : S \to C$ with fibers isomorphic to $\mathbb{P}^1_{\mathbb{C}}$. The bundle $p^*(E)$ on S has a sub-line bundle $\mathcal{T}$ whose fiber over a point $s \in S$ is the line in $E_{p(s)}$ corresponding to s. The bundle $\mathcal{O}_S(1)$ is defined by the short exact sequence

$$0 \to \mathcal{T} \to p^*E \to \mathcal{O}_S(1) \to 0.$$

Prove that:

(a) $\mathrm{Pic}(S) = p^* \mathrm{Pic}(C) \oplus \mathbb{Z} \cdot \mathcal{O}_S(1)$.

(b) $\mathrm{H}^2(S, \mathbb{Z}) = \mathbb{Z} \cdot \mathrm{c}_1(\mathcal{O}_S(1)) \oplus \mathbb{Z} \cdot [F]$, where $[F]$ is the class of a fiber of p.

(c) $\mathrm{c}_1(\mathcal{O}_S(1))^2 = \deg(E)$.

(d) $\mathrm{c}_1(K_S) = -2\,\mathrm{c}_1(\mathcal{O}_S(1)) + (\deg(E) + 2g(C) - 2) \cdot [F]$.

Aufgabe 2 (4 Punkte)

A K3-surface S is a complex compact smooth surface with

$$\Omega^2_S \cong \mathcal{O}_S \text{ and } \mathrm{H}^1(S, \mathcal{O}_S) = 0.$$

(a) Prove that a smooth intersection of n-generic divisors $D_i \in H^0(\mathbb{P}^{n+2}, \mathcal{O}_{\mathbb{P}^{n+2}}(d_i))$ is a K3-surface if and only if $\sum_{i=1}^n d_i = n + 3$.

(b) Show that on a K3-surface there are no non-trivial torsion line bundles.

Aufgabe 3 (4 Punkte)

Let S be a smooth compact complex surface.

(a) Prove that if H is an ample line bundle on S, then $H \cdot C > 0$ for all irreducible smooth curves $C \in \mathrm{Div}(S)$.

(b) Prove that if $D \in \mathrm{Div}(S)$ satisfies $D^2 > 0$, then for a large enough positive integer n we have $h^0(nD) > 0$ or $h^0(-nD) > 0$.

Abgabe: Zu Beginn der Übung um **14:15** Uhr am **Mittwoch, den 3. Juli**.