Goethe-Universität Frankfurt
Institut für Mathematik
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Komplexe Algebraische Geometrie II
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Übungsblatt 1

Let D be a divisor on a variety X and recall the different definitions of the *complete linear system*

$$\begin{aligned} |D| &= \mathbb{P}H^0(X, \mathcal{O}_X(D)) = \{D' \in \mathrm{Div}(X) \,:\, D' \sim D \text{ and } D \geq 0\} = \\ &= \mathbb{P}L(D) := \{f \text{ meromorphic function on } X \,:\, (f) + D \geq 0\}\,. \end{aligned}$$

Assume that D is effective and write $D = (s_D)$ for a section $s_D \in H^0(X, \mathcal{O}_C(D))$. The equivalences between the definitions above are given by

$$\begin{array}{ccccc} \mathbb{P}H^0(X, \mathcal{O}_X(D)) & \longleftrightarrow & \{D' \sim D \text{ and } D \geq 0\} & \longleftrightarrow & \mathbb{P}L(D) \\ [s] = [\lambda s] & \mapsto & (s) & \mapsto & [s/s_D] \\ [f \cdot s_D] & \leftarrow\!\shortmid & D' = D + (f) & \leftarrow\!\shortmid & [f] = [\lambda \cdot f] \end{array}$$

More generally, any linear subspace $V \subset |D|$ is called a *linear system*. We define the *base locus* of V as

$$\mathrm{Bs}(V) = \bigcap_{D' \in V} D' = \{x \in X \,:\, s(x) = 0 \text{ for all } s \in V \subset \mathbb{P}H^0(X, \mathcal{O}_X(D))\}\,.$$

Aufgabe 1 (5 Punkte)

Consider the affine curve in $\mathbb{C}^2$ given by equation $y^3 = x^5 - 1$ and the projection $\pi(x, y) = x$. Let X be the curve given by the unique smooth completion of the above affine curve and call $\pi : X \to \mathbb{P}^1$ the extension of the above projection. Set $p = \pi^{-1}(\infty)$ and $r_\alpha = (e^{2\pi i \cdot \alpha/5} : 0 : 1)$ for $\alpha = 0, \ldots, 4$.

(a) Find the ramification divisor of π and compute the genus of X.

Hint: The ramification divisor of a map π is $\sum_{p \in X}(e_p - 1) \cdot [p]$, where e_p is the ramification index of π at p.

(b) Establish the linear equivalences $3p \sim 3r_\alpha$, for $\alpha = 0, \ldots, 4$, and $\sum_{\alpha=0}^{4} r_\alpha \sim 5p$.

(c) Determine the space $H^0(X, K_X)$ of holomorphic differentials on X.

Hint: Find $|D|$ where $D = (dx)$.

(d) Describe the canonical map $\phi_{K_X} : X \to \mathbb{P}^3$ and determine the equations of its image.

(e) Using part (c) show that for $D = \sum_{\alpha=0}^{4} r_\alpha$ one has $h^0(X, K_X(-D)) = 1$.
Use Riemann-Roch to conclude that $h^0(X, \mathcal{O}(D)) = 3$.

Aufgabe 2 (3 Punkte)

Let D be a divisor on a curve X.

(a) Prove that D is base-point-free if and only if $h^0(X, D - [p]) = h^0(X, D) - 1$ for all points $p \in X$.

(b) Prove that D is very ample if and only if $h^0(X, D - [p] - [q]) = h^0(X, D) - 2$ for all points $p, q \in X$.

Aufgabe 3 (4 Punkte)

Let D be a divisor on $\mathbb{P}^2$ and $V \subset |D|$ a linear system. Write $\varphi_V : \mathbb{P}^2 \dashrightarrow \mathbb{P}^N$ for its associated map and assume that the base locus $\mathrm{Bs}(V)$ consists of a single point P. Consider the blow-up $\pi : \mathrm{Bl}_P(\mathbb{P}^2) \to \mathbb{P}^2$ and prove that there exists a morphism $\Phi : \mathrm{Bl}_P(\mathbb{P}^2) \to \mathbb{P}^N$ everywhere defined such that $\Phi = \phi_V \circ \pi$.

Hint: What can you say about the linear systems

$$\pi^*V - kE = \{\pi^*D' - kE \,:\, D' \in V\}$$

for different $k \in \mathbb{Z}$, where $E = \pi^{-1}(P)$ is the exceptional divisor?

Aufgabe 4 (4 Punkte)

In this exercise we construct maps ϕ_L associated to certain line bundles L on projective spaces.

(a) Let L be a very ample line bundle on a variety X. Prove that $L^{\otimes m}$ is very ample for every $m \geq 1$.

(b) Consider the line bundle $\mathcal{O}(d)$ on $\mathbb{P}^n$. Recall that its global sections can be identified with homogeneous polynomials of degree d. Describe the map $\phi_{\mathcal{O}(d)}$, called the *Veronese embedding*. Why is it an embedding?

(c) Let $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ and write $p_i : X \to \mathbb{P}^{n_i}$ for the two natural projections. Describe the map ϕ_L associated to the line bundle $L = p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)$ on X, called the *Segre embedding*. What is the base locus $\mathrm{Bs}(|L|)$?

Abgabe Zu Beginn der Übung um **14:00** am **Mittwoch, den 8. Mai**.