Goethe-Universität Frankfurt
Institut für Mathematik
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Komplexe Algebraische Geometrie
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Übungsblatt 12

Aufgabe 1 (3 Punkte)

Let $(V, \langle\ ,\ \rangle, I)$ be an euclidean vector space with a compatible almost complex structure. Let L, Λ and H be respectively the Lefschetz operator (associated to I), its dual and the counting operator.
Show that the action of L, Λ and H defines a natural $\mathfrak{sl}(2,\mathbb{R})$-representation on $\bigwedge^* V^*$, i.e. a Lie algebra homomorphism $\mathfrak{sl}(2,\mathbb{R}) \to \mathrm{End}(\bigwedge^* V^*)$.
Hint: the Lie-algebra $\mathfrak{sl}(2,\mathbb{R})$ is the three-dimensional real vector space of all 2×2-matrices of trace zero, with Lie bracket defined as $[M,N] := MN - NM$. A basis is given by

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall that $[H,L] = 2L$, $[H,\Lambda] = -2\Lambda$ and $[L,\Lambda] = H$.

Aufgabe 2 (3 Punkte)

Consider the same setting as in the previous exercise. Verify that the condition $[L,\Lambda] = H$ can be generalized to

$$[L^j, \Lambda](\alpha) = j(k - n + j - 1)L^{j-1}(\alpha)$$

for all $\alpha \in \bigwedge^k V^*$, where $L^j = L \circ \cdots \circ L$ j-times.
Hint: use induction on j.

Aufgabe 3 (10 Punkte)

The aim of this exercise is to prove the following:
Theorem. *Let $(V, \langle\ ,\ \rangle, I)$ be an euclidean vector space of dimension $2n$ with a compatible almost complex structure and let L and Λ be the associated Lefschetz operators.*

(a) There exists a direct sum decomposition of the form:

$$\bigwedge^k V^* = \bigoplus_{i\geq 0} L^i(P^{k-2i}),$$

where $P^i := \{\alpha \in \bigwedge^i V^* \ :\ \Lambda\alpha = 0\}$. Moreover, this decomposition is othogonal with respect to $\langle\ ,\ \rangle$. This is the so-called "Lefschetz decomposition".

(b) If $k > n$, then $P^k = 0$.

(c) The map $L^{n-k} : P^k \to \bigwedge^{2n-k} V^*$ is injective for $k \leq n$.

(d) The map $L^{n-k} : \bigwedge^k V^* \to \bigwedge^{2n-k} V^*$ is bijective for $k \leq n$.

(e) If $k \leq n$, then $P^k = \{\alpha \in \bigwedge^k V^* \,|\, L^{n-k+1}\alpha = 0\}$.

Hint:
(a): since $\bigwedge^* V^*_{\mathbb{C}}$ is a finite-dimentional $\mathfrak{sl}(2,\mathbb{C})$-representation (see Aufgabe 1), it is a direct sum of irreducible ones (recall that a representation is *irreducible* if it has no proper subrepresentations).
Let $W \subset \bigwedge^* V^*_{\mathbb{C}}$ be an irreducible $\mathfrak{sl}(2,\mathbb{C})$-subrepresentation. Show that if $v \in W$ is an eigenvector of H with eigenvalue λ, then also Lv and Λv are eigenvectors of H, with eigenvalues $\lambda + 2$ and $\lambda - 2$ respectively.
Show that there exists an element $v \in W$ which is an eigenvector of H and such that $\Lambda v = 0$.
Show that W is generated by $v, Lv, L^2 v, \ldots$ and deduce the decomposition of $\bigwedge^k V^*$.
Using the (implicit) definition of Λ and Aufgabe 2, show that the previous decomposition is orthogonal.
(b): suppose that $\alpha \in P^k$ with $k > n$ and $i > 0$ minimal such that $L^i\alpha = 0$. Show (use Aufgabe 2) that $i = 0$. This means that $\alpha = 0$.
(c): suppose that $0 \neq \alpha \in P^k$ with $k \leq n$ and $i > 0$ minimal with $L^i\alpha = 0$. Show (use again Aufgabe 2) that $L^{n-k}\alpha \neq 0$.
(d): use the previous points.
(e): let $k \leq n$. Using the same idea as in (c), show that $P^k \subset \mathrm{Ker}(L^{n-k+1})$. Conversely, let $\alpha \in \bigwedge^k V^*$ with $L^{n-k+1}\alpha = 0$. Using the previous points, show that $L^{n-k+2}\Lambda\alpha = 0$, and in particular $\Lambda\alpha = 0$.

Abgabe Zu Beginn der Vorlesung um **10:00** am **Montag, den 4. Februar**.