Goethe-Universität Frankfurt
Institut für Mathematik
Wintersemester 2018
21. Januar 2019

Komplexe Algebraische Geometrie
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Übungsblatt 11

Aufgabe 1 (3 Punkte)

Let $B \subset \mathbb{C}^n$ be a polydisc and let $\alpha \in \mathcal{A}^{p,q}(B)$ be a d-closed form with $p, q \geq 1$. Show that there exists a form $\gamma \in \mathcal{A}^{p-1,q-1}(B)$ such that $\partial\bar{\partial}\gamma = \alpha$.
Hint: Show first that α is also d-exact. Let $k = p + q$ and let $\beta \in \mathcal{A}_{\mathbb{C}}^{k-1}(B)$ be such that $\mathrm{d}\beta = \alpha$. Study the decomposition of β in

$$\mathcal{A}_{\mathbb{C}}^{k-1}(B) = \bigoplus_{a+b=k-1} \mathcal{A}^{a,b}(B).$$

Aufgabe 2 (3 Punkte)

(a) Let $\omega = \frac{i}{2\pi} \sum \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i$ be the standard fundamental form on $\mathbb{C}^n$. Show that one can write $\omega = \frac{i}{2\pi}\partial\bar{\partial}\varphi$ for a positive function φ and determine φ. The function φ is called *Kähler potential.*

(b) Show that $\omega = \frac{i}{2\pi}\partial\bar{\partial}\log(|z|^2 + 1) \in \mathcal{A}^{1,1}(\mathbb{C})$ is the fundamental form of a compatible metric g that osculates to order two in any point.
Remark: this is the local shape of the Fubini-Study Kähler form of $\mathbb{P}^1$.

Aufgabe 3 (4 Punkte)

Let $(V, \langle\, ,\, \rangle)$ be an euclidean vector space and let I, J, K be compatible almost complex structures where $K = I \circ J = -J \circ I$. The associated fundamental forms are denoted by $\omega_I, \omega_J, \omega_K$.

(a) Show that V becomes in a natural way a vector space over the quaternions.
Hint: recall that the *quaternions* are the associative algebra $\mathbb{H} = \langle 1, i, j, k \rangle \cong \mathbb{R}^4$, with multiplication given by:

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \quad i^2 = j^2 = k^2 = -1.$$

(b) Show that $\omega_J + i \cdot \omega_K$ with respect to I is a form of type $(2, 0)$.

(c) How many natural almost complex structures do you see in this context?
Hint: let $a_1, a_2, a_3 \in \mathbb{R}$ and consider the endomorphism $W := a_1 I + a_2 J + a_3 K : V \to V$. When is W an almost complex structure?

Aufgabe 4 (6 Punkte)

Let X be a complex manifold of dimension n. The aim of this exercise is to make the tangent and cotangent bundles of X more explicit.

(a) Show that TX can be expressed as:

$$\pi : \bigsqcup_{x \in X} T^{1,0}_x X \longrightarrow X,$$

where $\pi(\mathbf{v}) := x$ if $\mathbf{v} \in T^{1,0}_x X$.
Hint: show that the cocycle description of this bundle is exactly the cocycle description of the tangent bundle. To do that, fix $\{(U_\alpha, \varphi_\alpha)\}$ a covering of local charts for X, and write $\varphi_\alpha = (z_1^\alpha, \ldots, z_n^\alpha)$ in local coordinates. Then consider as local trivializations the following maps:

$$\begin{aligned} \psi_\alpha : \pi^{-1}(U_\alpha) &\longrightarrow U_\alpha \times \mathbb{C}^n, \\ \sum_{j=1}^n v_j \cdot \frac{\partial}{\partial z_j^\alpha}\bigg|_x &\longmapsto \big(x, (v_1, \ldots, v_n)\big). \end{aligned}$$

(b) Follow the same idea as in the previous point and show that T^*X can be expressed as:

$$\pi : \bigsqcup_{x \in X} (T^*_x X)^{1,0} \longrightarrow X,$$

where $\pi(\omega) := x$ if $\omega \in (T^*_x X)^{1,0}$.
Hint: in this case define the trivializations:

$$\begin{aligned} \psi_\alpha : \pi^{-1}(U_\alpha) &\longrightarrow U_\alpha \times \mathbb{C}^n, \\ \sum_{j=1}^n v_j \cdot \mathrm{d}z_j^\alpha|_x &\longmapsto \big(x, (v_1, \ldots, v_n)\big). \end{aligned}$$

Abgabe Zu Beginn der Vorlesung um **10:00** am **Montag, den 28. Januar**.