

TORIC NEWTON–OKOUNKOV FUNCTIONS WITH AN APPLICATION TO THE RATIONALITY OF CERTAIN SESHADRI CONSTANTS ON SURFACES

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ABSTRACT. We initiate a combinatorial study of Newton–Okounkov functions on toric varieties with an eye on the rationality of asymptotic invariants of line bundles. In the course of our efforts we identify a combinatorial condition which ensures a controlled behavior of the appropriate Newton–Okounkov function on a toric surface. Our approach yields the rationality of many Seshadri constants that have not been settled before.

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1. INTRODUCTION

In the present paper, we start to develop methods to determine Newton–Okounkov functions in the case of toric varieties. As a by-product, we can show rationality of Seshadri constants for many new examples of toric surfaces.

Newton–Okounkov bodies are convex bodies which encode various facets of algebraic and symplectic geometry such as the local positivity of line bundles on varieties and going as far as geometric quantization [HHK20]. More specifically, it is possible to gain information on asymptotic invariants (Seshadri constants, pseudo-effective thresholds, Diophantine approximation constants) from well-chosen Newton–Okounkov bodies and concave (Newton–Okounkov-)functions on them.

Over the past decade, Newton–Okounkov theory has attracted a lot of attention. Many deep structural results have been proven, drawing information about varieties and their line bundles from Newton–Okounkov bodies. At the same time it also

became apparent that it is often very difficult to obtain precise information about Newton–Okounkov bodies and Newton–Okounkov functions in concrete cases.

Newton–Okounkov Functions.

Newton–Okounkov functions are concave functions on Newton–Okounkov bodies arising from multiplicative filtrations on the section ring of a line bundle. They have proven to be more evasive than Newton–Okounkov bodies themselves.

The first definition of Newton–Okounkov functions (in other terminology, concave transforms of multiplicative filtrations) in print is due to Boucksom–Chen [BC11]. These functions on the Newton–Okounkov body yield refined information [BKMS15, DKMS16b, Fuj16, KMS12, KMR19, MR15] about the arithmetic and the geometry of the underlying variety.

Already the most basic invariants of Newton–Okounkov functions contain highly non-trivial information. Perhaps the most notable example is the average of such a function — called the β -invariant of the line bundle and the filtration —, which is closely related to Diophantine approximation [MR15], and K-stability [Fuj16]. By the connection of the β -invariant to Seshadri constants [KMR19], its rationality could decide Nagata’s conjecture [DKMS16a]. Not surprisingly, concrete descriptions of these functions are very hard to obtain.

A structure theorem of [KMR19] identifies the subgraph of a Newton–Okounkov function coming from a geometric situation as the Newton–Okounkov body of a projective bundle over the variety in question. (Compare §4.1.1 below.)

Based on earlier work of Donaldson [Don02], Witt-Nyström [WN12] made the observation that the Newton–Okounkov function coming from a fully toric situation (meaning all of the line bundle, admissible flag, and filtration are torus-invariant) is piecewise affine linear with rational coefficients on the underlying Newton–Okounkov body, which happens to coincide with the appropriate moment polytope. In this very special situation, the function is in fact linear. (Compare Proposition 4.1 below.)

The next interesting case arises when we keep the toric polytope (that is, we work with a torus-invariant line bundle and a torus-invariant admissible flag), but we consider the order of vanishing at a general point to define the function. To our knowledge, no such function has been computed for toric varieties other than projective space.

While the usual dictionary between geometry and combinatorics is very effective in explaining torus-invariant geometry, when it comes to non-torus invariant phenomena, one does need the more general framework of Newton–Okounkov theory. Broadly speaking Newton–Okounkov theory would be toric geometry without a torus action; in more technical terms Newton–Okounkov theory replaces the natural gradings on cohomology spaces by filtrations.

In determining Newton–Okounkov functions in a not completely toric setting, our first goal is to devise a strategy to determine Newton–Okounkov functions associated to orders of vanishing on toric surfaces, and to apply it to interesting examples. The trick is to avoid blowing up the valuation point, which could result in losing control of the Mori cone.

Instead, we change the flag defining the Newton–Okounkov polytope to one which contains the valuation point and show that there is a piecewise linear transformation of the moment polytope into the new Newton–Okounkov polytope (see Corollary 3.8). This is reminiscent of the transformation constructed in [EH]. But the connection is, as of yet, unclear.

We can then employ arguments from convex geometry to provide upper and lower bounds for the desired function. We study combinatorial conditions which guarantee that the obtained upper and lower bounds agree.

In the case of anti-blocking polyhedra in the sense of Fulkerson [Ful71, Ful72] we obtain a particularly easy answer. Nevertheless, the strategy works much more generally.

Theorem A (Newton–Okounkov functions on toric surfaces). Let X be a smooth projective toric surface, D an ample divisor and Y_\bullet an admissible torus-invariant flag on X so that the Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ is anti-blocking.

Let Y'_\bullet be a torus-invariant flag opposite to the origin. Then the Newton–Okounkov function φ_R on $\Delta_{Y'_\bullet}(D)$ coming from the geometric valuation ord_R in a general point $R \in X$ is linear with integral slope.

Along the way, we formulate and prove existence and uniqueness of Zariski decomposition on toric surfaces in the language of polyhedra. One can “see” the decomposition in terms of the polygons.

Local Positivity.

Newton–Okounkov bodies and the Newton–Okounkov functions defined on them reveal a lot about positivity properties of the underlying line bundles. Just like in the toric case, one can use this convex geometric information to decide for instance if the line bundle whose Newton–Okounkov bodies we consider is ample or nef [KL17a, KL17b].

Using Newton–Okounkov theory one can even obtain localized information about line bundles. A line bundle is called positive or ample at a point of our variety [KL18a] if global sections of a high enough multiple yield an embedding of an open neighborhood of the point. Local positivity can be decided via Newton–Okounkov bodies [KL17a, KL17b, Roé16]. Even more, we can measure how positive the line bundle in question is [KL17a].

Local positivity is traditionally measured by Seshadri constants [KL18a, Laz04]. Originally invented by Demailly [Dem92] to attack Fujita’s conjecture on global generation, Seshadri constants have become the main numerical asymptotic positivity invariant (cf. [Laz04, Chapter 5],[Bau99]). While there has been considerable interest in this invariant’s behavior, many of its properties are still shrouded in mystery [Sze12].

One interesting question about Seshadri constants is if they are always rational numbers. This is widely believed to be false, but there has only been sporadic progress towards this issue. On surfaces, the rationality of Seshadri constants would imply the failure of Nagata’s conjecture [DKMS16a].

The rationality of Seshadri constants and related asymptotic invariants often follows from finite generation of appropriate a multi-graded ring or semigroup [ELM⁺06, CL12]. But finite generation questions tend to be wide open, and are typically skew to the major finite generation theorems of birational geometry.

In the literature around Newton–Okounkov bodies the involved valuation semi-groups are frequently assumed, from the outset, to be finitely generated (see [HK15, KM19], this is to obtain a toric degeneration as in [And13]). However, deciding finite generation of multigraded algebras or semigroups arising from a geometric setting is an utterly hard question.

We obtain results on the rationality of Seshadri constants in general points of toric surfaces using asymptotic considerations and convexity to circumvent some of these difficulties.

Previously, Lundman [Lun20] and Sano [San14] have verified rationality of these same Seshadri constants for restricted classes of (line bundles on) toric surfaces. As it turns out, a condition we call “zonotopally well-covered”, is sufficient to guarantee rationality of the Seshadri constant, and conjecturally also guarantees the success of our strategy for the Newton–Okounkov function.

Theorem B (Rationality of certain Seshadri constants). Let X be a smooth projective toric surface and D an ample divisor. If there is a torus-invariant flag Y_\bullet on X , and a primitive direction $v \in \mathbb{Z}^2$ so that $\Delta_{Y_\bullet}(D)$ is zonotopally well-covered with respect to v , then the Seshadri constant of (X, D) at a general point of X is rational.

This theorem reproves some of the cases covered in [Lun20, San14] and adds many new cases, even some where we can only conjecture what the Newton–Okounkov function looks like. We would like to point out that it was Ito in [Ito13, Ito14], who first studied the connection between Seshadri constants and Newton–Okounkov bodies (partially in the toric setting as well), however, he is mostly concerned with non-trivial lower bounds on Seshadri constants and so his work yields no rationality results.

It is worth mentioning at this point that certain pairs of subgraphs of Newton–Okounkov functions associated to torus-invariant and non-torus-invariant flags happen to be equidecomposable (cf. Remark 4.9). This is an exciting and unexpected phenomenon with possible ties to the mutations studied in [CFK⁺17]. We offer a conjectural explanation for this phenomenon.

Organization of the Paper

We start in Section 2 by fixing notation and giving necessary background information. Since our work sits on the fence between two areas, we give ample information on both. Section 3 is devoted to a self-contained combinatorial proof of Zariski decomposition on toric surfaces. In Section 4 we give description of Newton–Okounkov functions/concave transforms in the two relevant cases: when every actor is torus-invariant (Subsection 4.1) and when we are looking at the order of vanishing filtration coming from a general point (Subsection 4.2). The latter part contains a proof of Theorem A and the outline of our general strategy. Section 5 contains the application of our results on Newton–Okounkov functions to the rationality of Seshadri constants.

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2. BACKGROUND AND NOTATION

We work over an algebraically closed field K . Although no arguments depend on characteristic zero, for convenience we will assume $K = \mathbb{C}$.

2.1. Toric Varieties. Since we will mostly consider the case of toric varieties, we review some basic results and fix notation regarding toric varieties and divisors in particular the interplay between algebraic geometry and combinatorics. We will mainly follow the conventions used in [CLS11] which gives a broad introduction to toric varieties.

Let X be an n -dimensional smooth projective *toric* variety. Then $X = X_\Sigma$ is determined by a complete unimodular fan Σ in $N_\mathbb{R} = N \otimes \mathbb{R} \simeq \mathbb{R}^n$, where $N \simeq \mathbb{Z}^n$ denotes the underlying lattice of one-parameter subgroups. Its dual, the lattice of characters is denoted by $M = \text{Hom}_\mathbb{Z}(N, \mathbb{Z})$ and the associated vector space by $M_\mathbb{R} = M \otimes \mathbb{R}$. We denote the underlying torus by $\mathbb{T} = N \otimes K^*$.

Let $\Sigma(i)$ denote the set of i -dimensional cones of the fan. Each ray $\rho \in \Sigma(1)$ is determined by a primitive ray generator $u_\rho \in N$. Since Σ is unimodular, the primitive ray generators of each maximal cone $\sigma \in \Sigma$ form a basis of N . The toric patches will be denoted by U_σ for $\sigma \in \Sigma$.

Since X is smooth, a divisor D is a Weil divisor if and only if it is a Cartier divisor, i.e. $\text{Pic}(X) = \text{Cl}(X)$. Due to the Orbit-Cone correspondence a ray $\rho \in \Sigma(1)$ gives a codimension-1 orbit whose closure $V(\rho)$ is a torus-invariant prime divisor on X which we denote by D_ρ . Let $K_X = -\sum_\rho D_\rho$ denote the canonical divisor on X .

Given a torus-invariant divisor $D = \sum_\rho a_\rho D_\rho$ on X , it determines a polyhedron

$$(1) \quad P_D := \{m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\},$$

which is actually a polytope, since Σ is complete. Denote the normal fan of P_D by Σ_{P_D} .

To a Cartier divisor D on X we can associate the sheaf $\mathcal{O}_X(D)$, which is the sheaf of sections of a line bundle $\pi: V_{\mathcal{L}} \rightarrow X$ which we will denote by \mathcal{L} for short. This is a one-to-one correspondence. For convenience we will use these terms interchangeably. Furthermore, we can describe a Cartier divisor $D = \sum_\rho a_\rho D_\rho$ in terms of its support function $\text{SF}_D: |\Sigma| \rightarrow \mathbb{R}$ which is linear on each $\sigma \in \Sigma$ with $\text{SF}_D(u_\rho) = -a_\rho$ for all $\rho \in \Sigma(1)$. Additionally, a Cartier divisor is determined by its Cartier data $\{m_\sigma\}_{\sigma \in \Sigma}$, where the m_σ satisfy $D|_{U_\sigma} = \text{div}(\chi^{-m_\sigma})|_{U_\sigma}$ for all $\sigma \in \Sigma$. The vector space of global sections arises from the characters for the lattice points inside the polytope, namely

$$(2) \quad \Gamma(X, \mathcal{O}_X(D)) = \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m.$$

A lot of properties of the divisor D can be read from the polytope P_D . A divisor D is big if and only if its associated polytope P_D is full-dimensional and ample if and only if the normal fan Σ_{P_D} of the polytope P_D is exactly Σ . It is nef if and only if all defining inequalities of P_D are tight and the fan Σ is a refinement of the normal fan Σ_{P_D} .

Let D be an ample divisor on X with corresponding polytope P_D . Then for each vertex $Q \in \text{vert}(P_D)$ the primitive vectors m_1^Q, \dots, m_n^Q in its adjacent edge directions are a lattice basis for the lattice M and therefore specify an associated coordinate system of the space $M_{\mathbb{R}}$.

For a strongly convex rational polyhedral cone τ in $N_{\mathbb{R}}$, let N_τ be the sublattice of N spanned by points in $N \cap \tau$, then we denote the quotient lattice by $N(\tau) = N/N_\tau$. Let Σ be a fan in $N_{\mathbb{R}}$ and $\tau \in \Sigma$. We consider the quotient map $N_{\mathbb{R}} \rightarrow N(\tau)_{\mathbb{R}}$ and denote by $\bar{\sigma}$ the image of a cone $\sigma \in \Sigma$ containing τ . Then

$$\text{star}(\tau) := \{\bar{\sigma} \subseteq N(\tau)_{\mathbb{R}} \mid \tau \preceq \sigma \in \Sigma\}$$

is a fan in $N(\tau)_{\mathbb{R}}$.

There is again a toric variety associated to this fan. Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope with normal fan Σ_P and associated toric variety X_P . Then each face $Q \preceq P$ corresponds to a cone $\sigma_Q \in \Sigma_P$. According to Propositions 3.2.7 and 3.2.9 in [CLS11] we obtain the isomorphisms

$$X_{\text{star}(\sigma_Q)} \simeq V(\sigma_Q) \simeq X_Q$$

between the resulting varieties, where X_Q is the variety that is associated to the lattice polytope Q .

2.2. Measuring Polytopes. Let $P \subseteq M_{\mathbb{R}}$ be a polytope. The *width of P with respect to a linear functional $u \in N$* is defined as

$$\text{width}_u(P) := \max_{Q, T \in P} |u(Q) - u(T)|.$$

For a rational line segment L there is the notion of *lattice length*, denoted by $\text{length}_M(L)$. Let therefore L be the segment connecting the rational points $Q, T \in M_{\mathbb{Q}}$ and denote by $m \in M$ the shortest lattice vector on the ray spanned by $Q - T$. Then we define $\text{length}_M(L) := |j|$, where $j \in \mathbb{Q}$ such that $Q - T = jm$.

2.3. Newton–Okounkov Bodies. The rich theory of toric varieties provides a very useful dictionary between algebraic geometry and convex geometry. It turned out that there is a one-to-one correspondence between the following sets.

$$\{P \subseteq M_{\mathbb{R}} \mid P \text{ is a full-dimensional lattice polytope}\}$$

\leftrightarrow

$$\{(X_{\Sigma}, D) \mid \Sigma \text{ is a complete fan in } N_{\mathbb{R}}, D \text{ a torus-invariant ample divisor on } X_{\Sigma}\}$$

This allows us to translate questions about algebraic geometric properties of the pair (X_{Σ}, D) into questions about P_D on the polytopal side and the other way round. In the 90's in [Oko96] Okounkov laid the groundwork to generalize this idea to arbitrary projective varieties motivated by questions coming from representation theory. Based on that Lazarsfeld–Mustașă [LM09] and Kaveh–Khovanskii [KK12] independently developed a systematic theory of Newton–Okounkov bodies about ten years later. It lets us assign a convex body to a given pair (X, D) that captures much of the asymptotic information about its geometry.

We review the construction of Newton–Okounkov bodies and will hereby mostly follow the approach and notation in [LM09].

Let X be an irreducible projective variety of dimension n . We fix an admissible flag

$$Y_{\bullet}: X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n$$

of irreducible subvarieties, where *admissible* requires that $\text{codim}_X Y_i = i$ for all $0 \leq i \leq n$ and that Y_i is smooth at the point Y_n for all $0 \leq i \leq n$.

Additionally let D be a big Cartier divisor on X and $\mathcal{L} = \mathcal{O}_X(D)$ the associated big line bundle.

We define a valuation-like map

$$\begin{aligned} \text{val} = \text{val}_{Y_\bullet} : \Gamma(X, \mathcal{O}_X(kD)) \setminus \{0\} &\rightarrow \mathbb{Z}^n \\ s &\mapsto (\text{val}_1(s), \dots, \text{val}_n(s)) \end{aligned}$$

for any $k \in \mathbb{N}$ in the following way. Given a non-zero global section $s \in \Gamma(X, \mathcal{O}_X(kD))$ set

$$\text{val}_1 = \text{val}_1(s) := \text{ord}_{Y_1}(s).$$

More explicitly, since $Y_0 = X$ is smooth at Y_n , there exists an open neighborhood U_0 of Y_n in which Y_1 is a Cartier divisor. Let f_1 denote its locally defining regular function and let g_1 be the regular function that locally defines s . Then $\text{ord}_{Y_1}(s)$ is the maximal integer j , such that f_1^j divides g_1 . This determines a section

$$\tilde{s}_1 \in \Gamma(X, \mathcal{O}_X(kD - \text{val}_1 Y_1))$$

which does not vanish identically along Y_1 . Therefore its restriction yields a non-zero section

$$(3) \quad s_1 \in \Gamma(Y_1, \mathcal{O}_{Y_1}(kD - \text{val}_1 Y_1))$$

that is locally given as $\frac{g_1}{f_1^{\text{val}_1}}|_{Y_1}$ in terms of regular functions. Next choose a suitable open set U_1 on Y_1 and set

$$\text{val}_2(s) := \text{ord}_{Y_2}(s_1)$$

in the same manner. Proceed iteratively to determine val by the successive orders of vanishing along the subvarieties Y_i , i.e.

$$\text{val}_i(s) := \text{ord}_{Y_i}(s_{i-1}).$$

Having this map we want to associate a convex body to the given data.

Definition 2.1. *The Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ of D (with respect to the flag Y_\bullet) is defined to be the set*

$$\Delta_{Y_\bullet}(D) := \overline{\bigcup_{k \geq 1} \frac{1}{k} \{ \text{val}_{Y_\bullet}(s) \mid s \in \Gamma(X, \mathcal{O}_X(kD)) \setminus \{0\} \}} \subseteq \mathbb{R}^n.$$

By definition the Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ is a convex set. Lemma 1.11 in [LM09] states that it is also a bounded subset and therefore compact and by Proposition 4.1 in [LM09] it only depends on the numerical equivalence class of D . For its volume we have $n! \cdot \text{vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(D)) = \text{vol}(D)$ independently of the flag Y_\bullet by Theorem 2.3 in [LM09].

2.3.1. Newton–Okounkov Bodies on Surfaces

Given the data X , Y_\bullet and D there is no straight forward way to compute the corresponding Newton–Okounkov body that works in general. For the case of surfaces the existence of Zariski decomposition plays the key role for a promising approach. In its original form it goes back to Zariski [Zar62], where he gave a way to uniquely decompose a given effective \mathbb{Q} -divisor D into a positive part D^+ and a negative

part D^- . This result was reproved by Bauer [Bau09] and also Fujita provided an alternative proof in [F⁺79] which also extends to pseudo-effective \mathbb{R} -divisors. Here we review the statement in the most general form.

Theorem 2.2 ([KMM87], Theorem 7.3.1). *Let D be a pseudo-effective \mathbb{R} -divisor on a smooth projective surface. Then there exists a unique effective \mathbb{R} -divisor*

$$D^- = \sum_{i=1}^{\ell} a_i N_i$$

such that

- (1) $D^+ = D - D^-$ is nef,
- (2) D^- is either zero or its intersection matrix $(N_i \cdot N_j)_{i,j}$ is negative definite,
- (3) $D^+ \cdot N_i = 0$ for $i \in \{1, \dots, \ell\}$.

Furthermore, D^- is uniquely determined as a cycle by the numerical equivalence class of D ; if D is a \mathbb{Q} -divisor, then so are D^+ and D^- . The decomposition

$$D = D^+ + D^-$$

is called the Zariski decomposition of D .

The above Theorem provides a decomposition $D = D^+ + D^-$ for a given divisor D . To receive information about the shape of the resulting Newton–Okounkov body it is important to know how that decomposition varies once we perturb the divisor. Let $C = Y_1$ denote the curve in the flag. Start at D , move in direction of $-C$ towards the boundary of the big cone $\text{Big}(X)$ and keep track of the variation of the Zariski decomposition of $D_t := D - tC$. For more details see [BKS04], also [KL18a, Chapter 2]. The next Theorem is a fundamental result that applies this procedure in order to compute the Newton–Okounkov body $\Delta_{Y_\bullet}(D)$.

Theorem 2.3 ([LM09], Theorem 6.4). *Let X be a smooth projective surface, D a big divisor (or more generally, a big divisor class), $Y_\bullet: X \supseteq C \supseteq \{z\}$ an admissible flag on X . Then there exist continuous functions $\alpha, \beta: [\nu, \mu] \rightarrow \mathbb{R}_{\geq 0}$ such that $0 \leq \nu \leq \mu =: \mu_C(D)$ are real numbers,*

- (1) ν = the coefficient of C in D^- ,
- (2) $\alpha(t) = \text{ord}_z D_t^-|_C$,
- (3) $\beta(t) = \alpha(t) + (D_t^+ \cdot C)$.

Then the associated Newton–Okounkov body is given by

$$\Delta_{Y_\bullet}(D) = \{(t, m) \in \mathbb{R}^2 \mid \nu \leq t \leq \mu, \alpha(t) \leq m \leq \beta(t)\}.$$

Moreover, α is convex, β is concave and both are piecewise linear.

As an immediate consequence the Newton–Okounkov body will always be a polytope in \mathbb{R}^2 .

2.4. Functions on Newton–Okounkov Bodies Coming from Geometric Valuations. The construction of Newton–Okounkov functions in the sense of concave transforms of filtrations goes back to Boucksom–Chen [BC11] and Witt–Nyström [WN14] who introduced them from different perspectives and in a more general way than considered in the following. We will focus on functions coming from geometric valuations as dealt with in [KMS12] and recall the definition restricted to that case.

Given an irreducible projective variety X , an admissible flag Y_\bullet and a big divisor D let $\Delta_{Y_\bullet}(D)$ be the corresponding Newton–Okounkov body. Let now $Z \subseteq X$ be a smooth irreducible subvariety. Then we define a Newton–Okounkov function φ_Z in a two-step process. A point $m \in \Delta_{Y_\bullet}(D)$ is called a *valuative* point, if

$$m \in \text{Val}_{Y_\bullet} := \bigcup_{k \geq 1} \frac{1}{k} \{ \text{val}_{Y_\bullet}(s) \mid s \in \Gamma(X, \mathcal{O}_X(kD)) \setminus \{0\} \}.$$

For a valuative point $m \in \Delta_{Y_\bullet}(D)$ set

$$\begin{aligned} \tilde{\varphi}_Z: \text{Val}_{Y_\bullet} &\rightarrow \mathbb{R} \\ m &\mapsto \lim_{k \rightarrow \infty} \frac{1}{k} \sup \{ t \in \mathbb{R} \mid \text{it exists } s \in \Gamma(X, \mathcal{O}_X(kD)) \\ &\quad \text{with } \text{val}_{Y_\bullet}(s) = km, \text{ord}_Z(s) \geq t \}. \end{aligned}$$

Due to Lemma 2.6 in [KMS12] the set of valuative points Val_{Y_\bullet} is dense in $\Delta_{Y_\bullet}(D)$. For all non-valuative points $m \in \Delta_{Y_\bullet}(D) \setminus \text{Val}_{Y_\bullet}$ set $\tilde{\varphi}_Z(m) := 0$. To define a meaningful function on the whole Newton–Okounkov body we use the concave envelope.

Definition 2.4. Let $\Delta \subseteq \mathbb{R}^n$ be a compact convex set and $f: \Delta \rightarrow \mathbb{R}$ a bounded real-valued function on Δ . The closed convex envelope f^c of f is defined as

$$f^c := \inf \{ g(x) \mid g \geq f \text{ and } g: \Delta \rightarrow \mathbb{R} \text{ is concave and upper-semicontinuous} \}.$$

Definition 2.5. Define the Newton–Okounkov function φ_Z coming from the geometric valuation associated to Z as

$$\begin{aligned} \varphi_Z: \Delta_{Y_\bullet}(D) &\rightarrow \mathbb{R} \\ m &\mapsto \tilde{\varphi}_Z^c(m). \end{aligned}$$

Due to Lemma 4.4 in [KMS12] taking the concave envelope does not effect the values of the underlying function $\tilde{\varphi}_Z(m)$ for valuative points $m \in \text{Val}_{Y_\bullet}$.

Computing the actual values of a Newton–Okounkov function φ_Z becomes extremely difficult and thus the functions are not well-known even in some of the easiest cases. In general as far as the formal properties of φ_Z we will make use of the following know facts.

- φ_Z is non-negative and concave ([WN14] or [BC11], Lemma 1.6, 1.7).

- φ_Z depends only on the numerical equivalence class of D ([KMS12], Proposition 5.6).
- φ_Z is continuous if $\Delta_{Y_\bullet}(D)$ is a polytope ([KMS12], Theorem 1.1).
- The numbers

$$\max_{\Delta_{Y_\bullet}(D)} \varphi_Z \quad \text{and} \quad \int_{\Delta_{Y_\bullet}(D)} \varphi_Z$$

are independent of the choice of Y_\bullet . ([DKMS16b], Theorem 2.4, [BC11], Corollary 1.13).

In our setting an observation made in Lemma 1.4.10 in [KL18b] has the following immediate consequence. Consider the Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ with respect to the admissible flag $Y_\bullet: X \supseteq Y_1 \supseteq \cdots \supseteq Y_n$ and the big divisor D . Whenever we consider the function φ_{Y_n} coming from the geometric valuation ord_{Y_n} of the point Y_n in the flag Y_\bullet , then it is bounded from above by the sum of coordinates, i.e.

$$(4) \quad \begin{aligned} \varphi_{Y_n}: \Delta_{Y_\bullet}(D) &\rightarrow \mathbb{R} \\ (m_1, \dots, m_n) &\mapsto \varphi_{Y_n}(m_1, \dots, m_n) \leq m_1 + \cdots + m_n. \end{aligned}$$

There are plenty of examples for which the inequality in 4 is strict.

3. ZARISKI DECOMPOSITION FOR TORIC VARIETIES IN COMBINATORIAL TERMS

Given a smooth projective toric variety X , a torus-invariant flag Y_\bullet and a big divisor D , then the construction of the Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ recovers the polytope P_D by Proposition 6.1 in [LM09].

Let $D_{\rho_1}, \dots, D_{\rho_s}$ denote the torus-invariant prime divisors. Since the flag Y_\bullet is torus-invariant, we can assume an ordering of the divisors such that the subvarieties in the flag are given as $Y_i = D_{\rho_1} \cap \cdots \cap D_{\rho_i}$ for $1 \leq i \leq n$.

The underlying fan Σ is smooth and thus the primitive ray generators $u_{\rho_1}, \dots, u_{\rho_n}$ span a maximal cone σ and form a basis of the lattice N . This gives an isomorphism $N \simeq \mathbb{Z}^n$ and the dual isomorphism is given by

$$(5) \quad \begin{aligned} \Phi: M &\rightarrow \mathbb{Z}^n \\ m &\mapsto (\langle m, u_{\rho_i} \rangle)_{1 \leq i \leq n}, \end{aligned}$$

which extends linearly to the map $\Phi_{\mathbb{R}}: M_{\mathbb{R}} \xrightarrow{\simeq} \mathbb{R}^n$.

The Newton–Okounkov body remains the same if one changes it within its linear equivalence class. Hence we can assume $D|_{U_\sigma} = 0$, i.e. that if the divisor is given as $D = \sum_{\rho} a_{\rho} D_{\rho}$, then we have $a_{\rho} = 0$ for all $\rho \in \sigma(1)$.

This can also be interpreted in terms of Newton polytopes. For convenience we assume D to be ample. Each divisor D_{ρ} corresponds to a facet F_{ρ} of P_D and

all facets $F_{\rho_1}, \dots, F_{\rho_n}$ intersect in a vertex Q_σ that is associated to σ . Assuming $D|_{U_\sigma} = 0$ on the polytope side means to embed the polytope P_D in \mathbb{R}^n such that the vertex Q_σ is translated to the origin.

Let $s \in \Gamma(X, \mathcal{O}_X(D))$ be a global section with Newton polytope $\text{NP}(s) \subseteq P_D$. Then the order of vanishing of s along $Y_1 = D_{\rho_1}$ is given by the minimal lattice distance to F_{ρ_1} , that is

$$(6) \quad \text{ord}_{Y_1}(s) = \min_{m \in \text{NP}(s)} \langle u_{\rho_1}, m \rangle.$$

Let $F_1 \preceq \text{NP}(s)$ denote the face of the Newton polytope $\text{NP}(s)$ that has minimal lattice distance to F_{ρ_1} . Then $\text{ord}_{Y_2}(s_1) = \min_{m \in F_1} \langle u_{\rho_2}, m \rangle$ and in general we have

$$(7) \quad \text{ord}_{Y_{i+1}}(s_i) = \min_{m \in F_i} \langle u_{\rho_{i+1}}, m \rangle$$

for $1 \leq i \leq n-1$. Thus the map val sends the section s to the point $m \in \text{NP}(s)$ whose coordinates are lexicographically the smallest among all points of the Newton polytope. A similar argument applies for $k \geq 1$. Thus we obtain $\Delta_{Y_\bullet}(D) \subseteq P_D$. Since in particular all the vertices $Q \preceq P_D$ correspond to respective global sections extending characters χ^Q , this yields $P_D \subseteq \Delta_{Y_\bullet}(D)$ and therefore $P_D \simeq \Delta_{Y_\bullet}(D)$.

For our convenience we identify P_D with its image under $\Phi_{\mathbb{R}}$.

As the Newton–Okounkov bodies only depend on the numerical equivalence class of D , we can and often want to choose a torus-invariant representative. If D is given by a defining local equation then there is a combinatorial way to find one.

Proposition 3.1. *Let X be a smooth projective toric variety with associated fan Σ and D a divisor on X that is given by the local equation f in the torus for some $f \in K(X) \setminus \{0\}$. Then $D' := \sum_{\rho \in \Sigma(1)} -a_\rho D_\rho$ with coefficients*

$$(8) \quad a_\rho := \min_{m \in \text{supp}(f)} \langle m, u_\rho \rangle$$

is a torus-invariant divisor that is linearly equivalent to D , where u_ρ is the primitive ray generator of the ray $\rho \in \Sigma(1)$.

Proof. Consider the Cox ring $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ which is graded by the class group $\text{Cl}(X)$, see Chapter 5 in [CLS11] for details. For a cone $\sigma \in \Sigma$ we denote by $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$ the associated monomial in S and by $S_{x^{\hat{\sigma}}}$ the localization of S at $x^{\hat{\sigma}}$. Applying Lemma 2.2 [Cox95] to the $\{0\}$ -cone $\sigma_0 \in \Sigma$ gives an isomorphism of rings

$$\mathbb{C}[M] = \mathbb{C}[\check{\sigma}_0 \cap M] \simeq (S_{x^{\hat{\sigma}_0}})_0,$$

where $x^{\hat{\sigma}_0} = \prod_\rho x_\rho$ and $(S_{x^{\hat{\sigma}_0}})_0$ is the graded piece of degree 0.

Given a lattice point $m \in M$ the character χ^m is homogenized to the monomial $x^{(m)} = \prod_{\rho} x_{\rho}^{\langle m, u_{\rho} \rangle}$ by the corresponding map $\theta: \mathbb{C}[M] \rightarrow (S_{x^{\hat{\sigma}_0}})_0$. Thus homogenizing $f = \sum_{m \in \text{supp}(f)} b_m \chi^m \in \mathbb{C}[M]$ yields

$$\tilde{f} = \theta(f) = \theta \left(\sum_{m \in \text{supp}(f)} b_m \chi^m \right) = \sum_{m \in \text{supp}(f)} b_m \prod_{\rho} x_{\rho}^{\langle m, u_{\rho} \rangle} = \frac{g}{\left(\prod_{\rho} x_{\rho} \right)^k}$$

for some homogeneous $g \in S$ and some $k \in \mathbb{N}_0$. We can rewrite this as

$$(9) \quad \tilde{f} = \frac{g}{\left(\prod_{\rho} x_{\rho} \right)^k} = \prod_{\rho} x_{\rho}^{a_{\rho}} \cdot h$$

for $h \in S$ coprime with $\prod_{\rho} x_{\rho}$ and uniquely determined $a_{\rho} \in \mathbb{Z}$.

Since \tilde{f} is homogeneous of degree 0, it gives a rational function on X and we have $0 \sim \text{div}(\tilde{f}) = \text{div} \left(\prod_{\rho} x_{\rho}^{a_{\rho}} \right) + \text{div}(h)$. On the torus the zero sets of f and h agree. Since h is coprime with x_{ρ} for all $\rho \in \Sigma(1)$, it has no zeros or poles along the boundary components. Altogether we have

$$D = \text{div}(h) \sim \text{div} \left(\prod_{\rho} x_{\rho}^{-a_{\rho}} \right) =: D'.$$

Then D' is torus-invariant by construction.

It remains to show, that the coefficients a_{ρ} as in (9) satisfy equation (8). To see that note that the homogenization of f consists of summands of the form $b_m \prod_{\rho} x_{\rho}^{\langle m, u_{\rho} \rangle}$, where we sum over $m \in \text{supp}(f)$. But h is an element of the Cox ring and it is supposed to be coprime with $\prod_{\rho} x_{\rho}$. Therefore to obtain the expression in (9) we have to bracket the factor x_{ρ}^j for j maximal that is a common factor of all the summands for each $\rho \in \Sigma(1)$. The maximal j is precisely

$$a_{\rho} = \min_{m \in \text{supp}(f)} \langle m, u_{\rho} \rangle$$

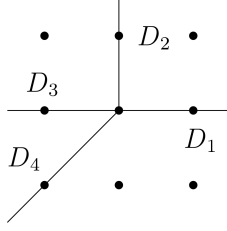
as claimed. □

We give an example to illustrate the proof of Proposition 3.1.

Example 3.2. We consider the Hirzebruch surface $X = \mathcal{H}_1$ associated to the fan in Figure 1, where the torus-invariant prime divisor D_i corresponds to the ray $\rho_i \in \Sigma(1)$ for $1 \leq i \leq 4$.

We work with the divisor

$$D = \overline{\{(x, y) \in \mathbb{T} \mid f(x, y) = xy^{-2} - 1 = 0\}}.$$

FIGURE 1. The fan Σ of the first Hirzebruch surface $X_\Sigma = \mathcal{H}_1$

Then the Cox-ring is given by $S = \mathbb{C}[x_1, x_2, x_3, x_4] = \mathbb{C}[x, y, x^{-1}, x^{-1}y^{-1}]$, where we write x_i for x_{ρ_i} . Homogenizing f yields

$$\begin{aligned} \theta(f) &= \sum_{m \in \text{supp}(f)} b_m \prod_{\rho} x_{\rho}^{\langle m, u_{\rho} \rangle} = x_1^1 x_2^{-2} x_3^{-1} x_4^1 - 1 \\ &= \frac{g}{\left(\prod_{\rho} x_{\rho}\right)^k} = \frac{x_1^3 x_3 x_4^3 - x_1^2 x_2^2 x_3^2 x_4^2}{(x_1 x_2 x_3 x_4)^2} \\ &= \prod_{\rho} x_{\rho}^{a_{\rho}} \cdot h = x_2^{-2} x_3^{-1} \cdot (x_1 x_4 - x_2^2 x_3), \end{aligned}$$

where the coefficients are $a_{\rho_1} = a_{\rho_4} = 0$, $a_{\rho_2} = -2$ and $a_{\rho_3} = -1$ and $h = x_1 x_4 - x_2^2 x_3$ is coprime with $x_1 x_2 x_3 x_4$. The same coefficients are obtained using Lemma 3.1.

$$\begin{aligned} a_{\rho_1} &= \min(\langle (0, 0), (1, 0) \rangle, \langle (1, -2), (1, 0) \rangle) = 0 \\ a_{\rho_2} &= \min(\langle (0, 0), (0, 1) \rangle, \langle (1, -2), (0, 1) \rangle) = -2 \\ a_{\rho_3} &= \min(\langle (0, 0), (-1, 0) \rangle, \langle (1, -2), (-1, 0) \rangle) = -1 \\ a_{\rho_4} &= \min(\langle (0, 0), (-1, -1) \rangle, \langle (1, -2), (-1, -1) \rangle) = 0. \end{aligned}$$

Thus $D' = \sum_{\rho \in \Sigma(1)} -a_{\rho} D_{\rho} = 2D_2 + D_3$ is a torus-invariant divisor which is linearly equivalent to D .

With Proposition 3.1 in hand, we can provide a combinatorial proof for the existence and uniqueness of Zariski decomposition for smooth toric surfaces independently of Theorem 2.2.

Theorem 3.3. *Let X be a smooth projective toric surface associated to the fan Σ and let D be a pseudo-effective torus-invariant \mathbb{R} -divisor on X . Then there exists a unique effective \mathbb{R} -divisor*

$$D^- = \sum_{i=1}^{\ell} c_i N_i$$

such that

$$(1) \ D^+ = D - D^- \text{ is nef,}$$

- (2) D^- is either zero or its intersection matrix $(N_i.N_j)_{i,j}$ is negative definite,
(3) $D^+.N_i = 0$ for $i \in \{1, \dots, \ell\}$.

If D is a \mathbb{Q} -divisor, then so are D^+ and D^- .

For the proof we will need the following Lemma.

Lemma 3.4. *Let X be the toric surface associated to the fan Σ . Let D_0, \dots, D_{k+1} be torus-invariant prime divisors with adjacent associated primitive ray generators $u_0, \dots, u_{k+1} \in \mathbb{R}^2$ such that $\text{cone}(u_0, u_{k+1})$ is pointed and $u_1, \dots, u_k \in \text{cone}(u_0, u_{k+1})$. Then*

$$(10) \quad \det((-D_i.D_j)_{1 \leq i, j \leq k}) = \det(u_0, u_{k+1}).$$

Proof. By Theorem 10.4.4 in [CLS11] the intersection numbers of the torus-invariant prime divisors D_1, \dots, D_k are given as

- $D_i.D_i = -\lambda_i$, where $u_{i-1} + u_{i+1} = \lambda_i u_i$
- and for $i \neq j$ as

$$D_i.D_j = \begin{cases} 1 & \text{if } \rho_i \text{ and } \rho_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

Thus the intersection matrix is of the form

$$(11) \quad A_k := (-D_i.D_j)_{1 \leq i, j \leq k} = \begin{pmatrix} \lambda_1 & -1 & 0 & \dots & \dots & 0 \\ -1 & \lambda_2 & -1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \dots & 0 \\ 0 & \dots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -1 & \lambda_{k-1} & -1 \\ 0 & \dots & \dots & 0 & -1 & \lambda_k \end{pmatrix}$$

We will prove by induction on k that (10) holds.

base case: For $k = 1$ we have

$$\begin{aligned} \det(u_0, u_2) &= u_0^{(1)}u_2^{(2)} - u_0^{(2)}u_2^{(1)} \\ &= \lambda_1 \left(u_0^{(1)}u_1^{(2)} - u_0^{(2)}u_1^{(1)} \right) \\ &= \lambda_1, \end{aligned}$$

since Σ is smooth. A similar computation applies for $k = 2$.

induction step: Let $k \geq 3$ be given and suppose (10) is true for all integers smaller than k . Note that the determinant of the tridiagonal matrix A_k fulfills a particular

recurrence relation, since it is an extended continuant. The recurrence relation is given by

$$\det(A_0) = 0, \det(A_1) = 1, \text{ and } \det(A_k) = \lambda_k \det(A_{k-1}) - \det(A_{k-2}).$$

Thus we have

$$\begin{aligned} \det(A_k) &= \lambda_k \det(A_{k-1}) - \det(A_{k-2}) \\ &\stackrel{\text{IH}}{=} \lambda_k \det(u_0, u_k) - \det(u_0, u_{k-1}) \\ &= \det(u_0, \lambda_k u_k - u_{k-1}) \\ &= \det(u_0, u_{k+1}) \end{aligned}$$

as claimed. \square

Proof of Theorem 3.3. Since the divisor D is torus-invariant, it is given as $D = \sum_{\rho} a_{\rho} D_{\rho}$. We can assume, that D is effective, i.e. $a_{\rho} \geq 0$ for all $\rho \in \Sigma(1)$. This defines the polygon

$$P_D = \{m \in M_{\mathbb{R}} \mid \langle m, u_{\rho} \rangle \geq -a_{\rho} \text{ for all } \rho \in \Sigma(1)\}.$$

Let $\tilde{a}_{\rho} \in \mathbb{R}$ be the coefficients, such that

$$P_D = \{m \in M_{\mathbb{R}} \mid \langle m, u_{\rho} \rangle \geq -\tilde{a}_{\rho} \text{ for all } \rho \in \Sigma(1)\}$$

and all the inequalities are tight on P_D , i.e. for all $\rho \in \Sigma(1)$ there exists a point $m \in P_D$ such that $\langle m, u_{\rho} \rangle = -\tilde{a}_{\rho}$.

Set $D^+ := \sum_{\rho} \tilde{a}_{\rho} D_{\rho}$ and $D^- := \sum_{\rho} (a_{\rho} - \tilde{a}_{\rho}) D_{\rho}$. Then

$$D = \sum_{\rho} a_{\rho} D_{\rho} = D^+ + D^-$$

and $(a_{\rho} - \tilde{a}_{\rho}) \geq 0$ by definition. We now show that this satisfies (1)–(3).

- (1) Since X is a surface, the divisor D^+ is nef by construction.
- (2) Let $D_{\rho} \cdot D_{\rho} \geq 0$ for some $\rho \in \Sigma(1)$. There exists a vector $v \in M$, such that v is orthogonal to $u_{\rho'}$ and $\langle v, u_{\rho} \rangle < 0$, where ρ' is a ray adjacent to ρ . Then the inequality corresponding to ρ is tight on P_D , i.e. $a_{\rho} = \tilde{a}_{\rho}$, because otherwise the polytope P_D would be unbounded in the direction of v . Thus only negative curves will appear in D^- .

The matrix $(N_i \cdot N_j)_{i,j}$ is negative definite, if all the leading principal minors of $(-N_i \cdot N_j)_{i,j}$ are positive. Label the negative curves that appear in the negative part D^- as $\{N_1, \dots, N_{\ell}\}$ in such a way that adjacent rays are given consecutive indices counter clockwise. Then the intersection matrix $(N_i \cdot N_j)_{i,j}$ is a block matrix, where each block is of the form (11) as in Lemma 3.4. So let $\{N_1, \dots, N_k\} \subseteq \{N_1, \dots, N_{\ell}\}$ be adjacent negative curves, that form a sub block $(-N_i \cdot N_j)_{1 \leq i,j \leq k}$ of the matrix $(-N_i \cdot N_j)_{1 \leq i,j \leq \ell}$ and

denote by C_0 and C_{k+1} the remaining curves whose rays are adjacent to ρ_1 and ρ_k as indicated in Figure 2.

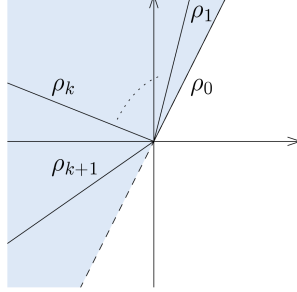


FIGURE 2. Adjacent rays $\rho_0, \dots, \rho_{k+1}$ of the prime divisors C_0, \dots, C_{k+1}

For the ray generators it holds that $u_1, \dots, u_k \in \text{cone}(u_0, u_{k+1})$ and that $\text{cone}(u_0, u_{k+1})$ is convex, for otherwise the polytope P_D would be unbounded in the direction of v' , where $v' \in M$ is chosen such that it is orthogonal to u_0 and $\langle v', u_{k+1} \rangle > 0$.

Thus it remains to show that the determinant of each such sub-block matrix is positive. According to Lemma 3.4 we have

$$\det(-N_i \cdot N_j)_{1 \leq i, j \leq k} = \det(u_0, u_{k+1}).$$

Let $u_0 = (m_1, m_2)$ and $u_{k+1} = (m'_1, m'_2)$ and assume without loss of generality that $m_1 > 0$. Since $\text{cone}(u_0, u_{k+1})$ is convex and $u_1, \dots, u_k \in \text{cone}(u_0, u_{k+1})$, it holds that $m'_2 > \frac{m_2}{m_1} m'_1$, because otherwise the polytope P_D would be unbounded. It follows that

$$\det(u_0, u_{k+1}) = \det \begin{pmatrix} m_1 & m'_1 \\ m_2 & m'_2 \end{pmatrix} = m_1 m'_2 - m'_1 m_2 > 0.$$

An similar argument works for $m_1 \leq 0$. Thus altogether we have that $(N_i \cdot N_j)_{i, j}$ is negative definite, since all sub block matrices of $(-N_i \cdot N_j)_{i, j}$ have a positive determinant.

- (3) Let $\rho \in \Sigma(1)$ be a ray for which D_ρ appears in the negative part D^- of the decomposition. Then by construction of D^+ its corresponding face $F_\rho \preceq P_{D^+}$ is a vertex. Using Proposition 6.3.8 in [CLS11] it follows that

$$D^+ \cdot D_\rho = |F_\rho \cap M| - 1 = 0,$$

when P_D is lattice polytope. A similar argument works in the non-integral case using $\text{length}_M(F_\rho)$.

The above gives the existence of a Zariski decomposition. It remains to show uniqueness of D^- . Assume we have a decomposition

$$D = \overline{D}^+ + \overline{D}^- = \sum_{\rho} \overline{a}_\rho D_\rho + \sum_{\rho} (a_\rho - \overline{a}_\rho) D_\rho.$$

Since \overline{D}^+ is supposed to be nef which translates into bar non tight inequalities for $P_{\overline{D}^+}$, we have $\bar{a}_\rho \leq \tilde{a}_\rho$ for all $\rho \in \Sigma(1)$. Let $\rho \in \Sigma(1)$ be the ray of a divisor D_ρ that appears in the negative part \overline{D}^- . Then as argued before this has to be a negative curve. But due to (3) the corresponding face F_ρ of $P_{\overline{D}^+}$ has to be a vertex and therefore it follows that $\bar{a}_\rho = \tilde{a}_\rho$. This yields uniqueness of \overline{D}^- . \square

Although we can always assume the divisor D to be torus-invariant, the shape of the Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ will heavily depend on the flag Y_\bullet which on the other hand is not necessarily torus-invariant. If the curve Y_1 in the flag is determined by an equation of the form $x^v - 1 = 0$ for some primitive $v \in \mathbb{Z}^2$, then we can show a combinatorial way to compute $\Delta_{Y_\bullet}(D)$.

Proposition 3.5. *Let X be a smooth projective toric surface, D a big divisor and $Y_\bullet: X \supseteq C \supseteq \{z\}$ an admissible flag on X , where the curve $C = \{x \in \mathbb{T} \mid x^v = 1\}$ for some primitive $v \in \mathbb{Z}^2$ and z a general smooth point on C . Then the associated function $\beta(t)$ in Theorem 2.3 is given by*

$$\begin{aligned} (12) \quad \beta(t) &= (D - tC)^+.C \\ (13) \quad &= (D - tC')^+.C \\ (14) \quad &= \text{MV}(P_{(D-tC)^+}, \text{NP}(x^v - 1)) \\ (15) \quad &= \text{MV}(P_D \cap (P_D + tv), \text{NP}(x^v - 1)) \end{aligned}$$

for $0 \leq t \leq \mu$, where C' is a torus-invariant curve that is linearly equivalent to C .

Proof. Since the Newton–Okounkov body only depends on the numerical equivalence class, we may assume that the divisor D is torus-invariant, i.e. $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$, where Σ is the fan associated to X with primitive ray generators u_ρ for $\rho \in \Sigma(1)$.

From Theorem 2.3 we know that $\beta(t) = (D - tC)^+.C$ for $\nu \leq t \leq \mu$, where $(D - tC)^+$ is the positive part of the Zariski decomposition of $D - tC$ and for C not part of $(D - tC)^-$ we have $\nu = 0$. Theorem 2.2 states that the decomposition is unique up to the numerical equivalence class of the given divisor. Let

$$C' = \sum_{\rho \in \Sigma(1)} - \min_{m \in \text{supp}(x^v - 1)} \langle m, u_\rho \rangle D_\rho$$

be the torus-invariant curve given as in Proposition 3.1. This means $C' \sim C$ and the curves are in particular numerically equivalent which yields (13). Due to Section 5.4/5.5 in [Ful93] the intersection product of two curves equals the mixed volume of the associated Newton polytopes and therefore we have (14).

To verify the remaining equality we show that

$$P_{(D-tC)^+} = P_D \cap (P_D + tv)$$

holds up to translation. For the torus-invariant curve $D - tC'$ the construction of its Zariski decomposition as in Theorem 3.3 guarantees the equality

$$P_{(D-tC')^+} = P_{(D-tC')}$$

for the corresponding polytopes. Consider its translation by tv , this gives

$$\begin{aligned} & P_{(D-tC')} + tv \\ &= \{m + tv \in M_{\mathbb{R}} \mid \langle m, u_{\rho} \rangle \geq -(a_{\rho} + t \cdot \min(0, \langle v, u_{\rho} \rangle)) \text{ for all } \rho \in \Sigma(1)\} \\ &= \{m \in M_{\mathbb{R}} \mid \langle m - tv, u_{\rho} \rangle \geq -(a_{\rho} + t \cdot \min(0, \langle v, u_{\rho} \rangle)) \text{ for all } \rho \in \Sigma(1)\} \\ &= \{m \in M_{\mathbb{R}} \mid \langle m, u_{\rho} \rangle \geq -a_{\rho} - t \cdot \min(0, \langle v, u_{\rho} \rangle) + t \langle v, u_{\rho} \rangle \text{ for all } \rho \in \Sigma(1)\} \\ &= \{m \in M_{\mathbb{R}} \mid \langle m, u_{\rho} \rangle \geq -a_{\rho} + \max\{0, t \langle v, u_{\rho} \rangle\} \text{ for all } \rho \in \Sigma(1)\}. \end{aligned}$$

On the other hand we have

$$P_D = \{m \in M_{\mathbb{R}} \mid \langle m, u_{\rho} \rangle \geq -a_{\rho} \text{ for all } \rho \in \Sigma(1)\}$$

and

$$P_D + tv = \{m \in M_{\mathbb{R}} \mid \langle m, u_{\rho} \rangle \geq -a_{\rho} + t \langle v, u_{\rho} \rangle \text{ for all } \rho \in \Sigma(1)\}.$$

Thus their intersection is the set

$$\begin{aligned} & P_D \cap (P_D + tv) \\ &= \{m \in M_{\mathbb{R}} \mid \langle m, u_{\rho} \rangle \geq -a_{\rho} \text{ and } \langle m, u_{\rho} \rangle \geq -a_{\rho} + t \langle v, u_{\rho} \rangle \text{ for all } \rho \in \Sigma(1)\} \\ &= \{m \in M_{\mathbb{R}} \mid \langle m, u_{\rho} \rangle \geq -a_{\rho} + \max(0, t \langle v, u_{\rho} \rangle) \text{ for all } \rho \in \Sigma(1)\} \end{aligned}$$

This verifies the equality in (15). \square

Example 3.6. We return to the Hirzebruch surface $X = \mathcal{H}_1$ from Example 3.2, and consider the big divisor $D = D_3 + 2D_4$ on X . Then for any admissible torus-invariant flag Y'_{\bullet} the associated Newton–Okounkov body $\Delta_D(Y'_{\bullet})$ coincides with a translate of the polytope P_D which can be seen in Figure 3.

We wish to determine the Newton–Okounkov body $\Delta_D(Y_{\bullet})$ given by a different flag $Y_{\bullet}: X \supseteq C \supseteq \{z\}$, where $C = \overline{\{(x, y) \in \mathbb{T} \mid y - 1 = 0\}}$ is a non-invariant curve, and z is a general smooth point on C . In local coordinates the curve C is given by the binomial $y - 1$ for $v = (0, 1)$ and thus has the line segment $\text{NP}(C) = \text{conv}((0, 0), (0, 1))$ as its Newton polytope. Using Lemma 3.1 we obtain the torus-invariant curve $C' = D_4$ which is linearly equivalent to C .

To determine the Newton–Okounkov body we use variation of Zariski decomposition for the divisor $D_t = D - tC$. To compute the upper part of the Newton–Okounkov body in terms of the piecewise linear function β , we move a copy of the polytope P_D in the direction of v as indicated in Figure 3.

The intersection $P_D \cap (P_D + tv)$ gives the polytope associated to $P_{(D-tC')^+}$. By Proposition 3.5 the function β is then given as

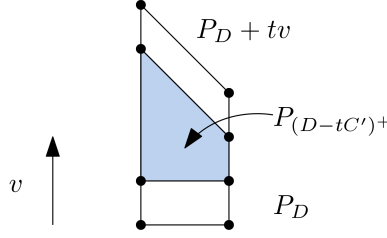


FIGURE 3. Moving a copy of P_D in the direction of v to obtain $P_D \cap (P_D + tv) = P_{(D-tC)^+}$

$$\begin{aligned}
 \beta(t) &= D_t^+ \cdot \mathcal{C} \\
 &= \text{MV}(P_D \cap (P_D + t \cdot (0, 1)), \text{NP}(y-1)) \\
 &= \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 2-t & \text{if } 1 \leq t \leq 2, \end{cases}
 \end{aligned}$$

where the mixed volume $\text{MV}(P_D \cap (P_D + t \cdot (0, 1)), \text{NP}(y-1))$ can be seen as the area of the shaded region in Figure 4.

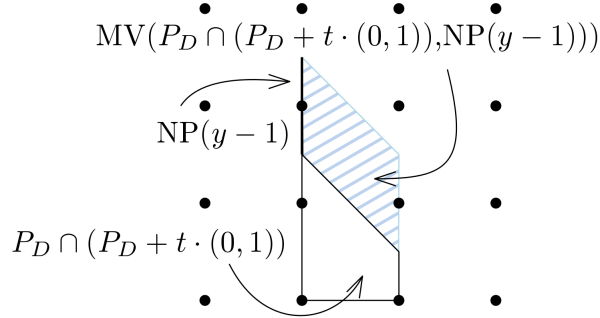


FIGURE 4. The mixed volume $\text{MV}(P_D \cap (P_D + t \cdot (0, 1)), \text{NP}(y-1))$

Since D is nef, we have $\nu = 0$ and since z can be chosen general enough on C , we also have $\alpha(t) \equiv 0$. Therefore the Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ is the polytope shown in Figure 5.

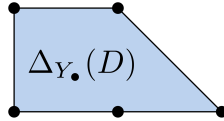


FIGURE 5. The Newton–Okounkov body $\Delta_{Y_\bullet}(D)$

Let us for simplicity assume, that $\nu = 0$ and that $\alpha \equiv 0$. Then the Newton-Okounkov body $\Delta_{Y_\bullet}(D)$ is completely determined by β .

Given the polytope P_D and the vector v the procedure described in Proposition 3.5 to compute the function β divides the polytope P_D into chambers. In the following we consider this process in detail. For that we introduce the following definition.

Definition 3.7. *Let $P \subseteq \mathbb{R}^2$ be a 2-dimensional polytope and let $v \in \mathbb{R}^2$ be a vector. Then we call a facet $F \preceq P$ sunny with respect to v , if $\langle v, u_F \rangle > 0$, where u_F is the inner facet normal of F . We call the set of all sunny facets of P with respect to v the sunny side of P with respect to v and denote it by $\text{sun}(P, v)$.*

Let $\text{sun}(P_D, v)$ be the sunny side of P_D with respect to v . By construction the function β is piecewise linear. There is a break point at time \tilde{t} , if and only if there exists a vertex $Q \in \text{vert}(P_D)$ such that

$$Q \in P_D \cap (\text{sun}(P_D, v) + \tilde{t}v).$$

Thus we move the sunny side $\text{sun}(P_D, v)$ along the polytope P_D in the direction of v . We start at time $t_0 = 0$. Whenever we hit a vertex $Q_i \in \text{vert}(P_D)$ at time t_i , we enter a new chamber as indicated in Figure 6.

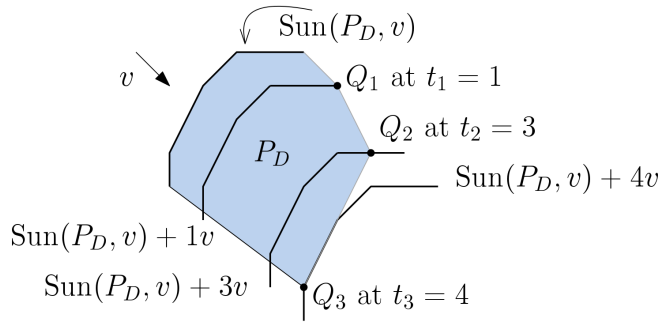


FIGURE 6. Break points Q_1, Q_2 and Q_3 of shifting the sunny side $\text{sun}(P_D, v)$ through P_D in the direction of v

Then $\beta(t)$ is linear in each time interval $[t_i, t_{i+1}]$ for $i \geq 0$.

The other part of the chamber structure comes from inserting a wall in the direction of v for each vertex $Q \in \text{sun}(P_D, v)$ and in the direction of $-v$ for each vertex $Q \in \text{sun}(P_D, -v)$ as it can be seen in Figure 7.

In the following we verify that for this particular chamber structure there exists a map between P_D and $\Delta_{Y_\bullet}(D)$ that is linear on each of the chambers.

For that we choose a coordinate system m_1, m_2 for $M_{\mathbb{R}} \simeq \mathbb{R}^2$, such that $v = (1, 0)$ without loss of generality. Consider the polytope $P_D \subseteq \mathbb{R}^2$ in (m_1, m_2) -coordinates

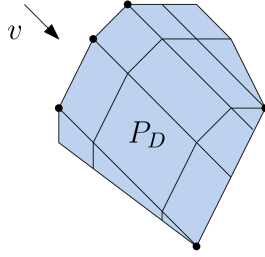


FIGURE 7. The chamber structure on P_D induced by the shifting process

and assume without loss of generality, that P_D lies in the positive orthant. We can write it as

$$P_D = \{(m_1, m_2) \in \mathbb{R}^2 \mid \gamma \leq m_2 \leq \delta, \ell(m_2) \leq m_1 \leq r(m_2)\},$$

for some $\gamma, \delta \in \mathbb{R}$ and some piecewise linear functions ℓ and r that determine the sunny sides $\text{sun}(P_D, v)$ and $\text{sun}(P_D, -v)$, respectively.

To determine the function β using the combinatorial approach from Proposition 3.5 we shift the sunny side $\text{sun}(P_D, v)$ through the polytope as depicted in Figure 6. Now we want to “tilt the polytope leftwards” such that the m_1 -coordinate of each point in the image expresses exactly the time, at which the point in the original polytope is visited in the shifting process. This is shown in Figure 8. To make this precise, map the polytope P_D via

$$\begin{aligned} \Psi_{\text{left}}: P_D \subseteq \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (m_1, m_2) &\mapsto (m_1 - \ell(m_2), m_2). \end{aligned}$$

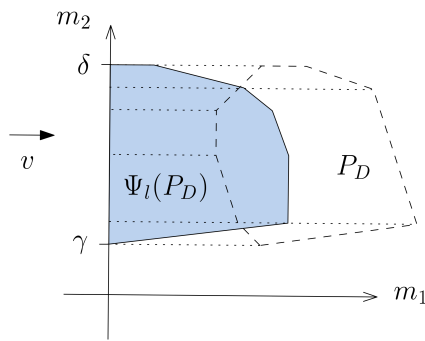


FIGURE 8. Tilting the polytope P_D leftwards via the map Ψ_{left}

By construction the map Ψ_{left} is a piecewise shearing of the original polytope and therefore volume preserving. Additionally $\Psi_{\text{left}}(P_D) \cap \{m_1 = t\}$ are exactly the images of the points of P_D that are visited at time t . Given $m_1 = t$ we now want to

determine $\beta(t)$. According to (15) it is given by

$$\begin{aligned}\beta(t) &= \text{MV}(P_D \cap (P_D + tv), \text{NP}(x^v - 1)) \\ &= \text{MV}(P_D \cap (\text{sun}(P_D, v) + tv), \text{NP}(x^v - 1)) \\ &= \text{MV}(\Psi_{\text{left}}(P_D) \cap \{m_1 = t\}, \text{NP}(x^v - 1)) \\ &= \text{length}_M(\Psi_{\text{left}}(P_D) \cap \{m_1 = t\}).\end{aligned}$$

The last equation holds, since v was chosen as $(1, 0)$. In a last step we want to “tilt the polytope downwards” similarly to the previous process as it can be seen in Figure 9. Therefore we can describe the polytope $\Psi_{\text{left}}(P_D)$ as

$$\begin{aligned}\Psi_{\text{left}}(P_D) &= \{(m_1, m_2) \in \mathbb{R}^2 \mid \gamma \leq m_2 \leq \delta, 0 \leq m_1 \leq r(m_2) - \ell(m_2)\} \\ &= \{(m_1, m_2) \in \mathbb{R}^2 \mid 0 \leq m_1 \leq \hat{\delta}, b(m_1) \leq m_2 \leq u(m_2)\},\end{aligned}$$

for some $\hat{\delta} \in \mathbb{R}$ and some piecewise linear functions b and u that determine the bottom and top of the polytope.

So set

$$\begin{aligned}\Psi_{\text{down}}: \Psi_{\text{left}}(P_D) \subseteq \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (m_1, m_2) &\mapsto (m_1, m_2 - b(m_1)).\end{aligned}$$

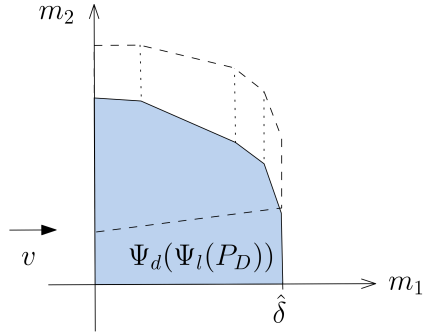


FIGURE 9. Tilting the polytope $\Psi_{\text{left}}(P_D)$ downwards via the map Ψ_{down}

By construction this is again a piecewise shearing of the polytope and therefore volume preserving. The image $\Psi_{\text{down}}(\Psi_{\text{left}}(P_D))$ is the subgraph of β and thus it coincides with the Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ with respect to the new flag Y_\bullet .

The above shows the following.

Corollary 3.8. *Let X be a smooth projective toric surface, D a big divisor and $Y_\bullet: X \supseteq C \supseteq \{z\}$ an admissible flag on X , where the curve C is given by a binomial $x^v - 1$ for a primitive $v \in \mathbb{Z}^2$ and z is a general smooth point on C . Then there exists a piecewise linear, volume preserving isomorphism $\Psi = \Psi_{\text{down}}(\Psi_{\text{left}})$ between the two Newton–Okounkov bodies P_D and $\Delta_{Y_\bullet}(D)$.*

4. NEWTON–OKOUNKOV FUNCTIONS ON TORIC VARIETIES

Whenever we determine the value of a Newton–Okounkov function $\varphi(m)$ for a point $m \in \Delta_{Y_\bullet}(D)$, we will often assume that m is a valutive point if not mentioned otherwise.

4.1. Completely Toric Case. In the case when all the given data is toric, we can completely describe the function φ_Z , and it even has a nice geometric interpretation. By “all data toric” we mean that X is a smooth toric variety, Y_\bullet is a flag consisting of torus-invariant subvarieties, D is a big torus-invariant divisor on X and $Z \subseteq X$ a torus-invariant subvariety.

In order to formulate and prove Proposition 4.1 below, we recall the combinatorics of the blow-up $\pi_Z: X^* \rightarrow X$ of Z . As Z is torus-invariant, it corresponds to a cone $\tau \in \Sigma$ of the fan. According to [CLS11, Definition 3.3.17] the fan Σ^* in $N_{\mathbb{R}}$ of the variety X^* is given by the star subdivision of Σ relative to τ : Set $u_\tau = \sum_{\rho \in \tau(1)} u_\rho$, $\rho_Z = \text{cone}(u_\tau)$, and for each cone $\sigma \in \Sigma$ containing τ , set

$$\Sigma_\sigma^*(\tau) = \{\sigma' + \rho_Z \mid \tau \not\subseteq \sigma' \subset \sigma\}$$

and the star subdivision of Σ relative to τ is the fan

$$\Sigma^* = \Sigma^*(\tau) = \{\sigma \in \Sigma \mid \tau \not\subseteq \sigma\} \cup \bigcup_{\sigma \supseteq \tau} \Sigma_\sigma^*(\tau).$$

Then the exceptional divisor E of the blow-up π_Z corresponds to the ray $\rho_Z \in \Sigma^*$, and the order of vanishing of a section s along Z is, by definition, the order of vanishing of $\pi_Z^*(s)$ along E .

The Cartier data $\{m_{\sigma^*}^*\}_{\sigma^* \in \Sigma^*(n)}$ of π_Z^*D is given by $m_{\sigma^*}^* = m_{\sigma^*}$ for $\sigma^* \in \Sigma(n)$ (i.e., $\sigma^* \not\supseteq \rho_Z$), and $m_{\sigma^*}^* = m_\sigma$ for $\sigma^* \in \Sigma_\sigma^*(\tau)(n)$.

Proposition 4.1. *Let X be an n -dimensional smooth projective toric variety associated to the unimodular fan Σ in $N_{\mathbb{R}}$. Furthermore let Y_\bullet be an admissible torus-invariant flag and D a big torus-invariant divisor on X with resulting Newton–Okounkov body $\Delta_{Y_\bullet}(D)$.*

Let $Z \subseteq X$ be an irreducible torus-invariant subvariety. Then the geometric valuation ord_Z yields a linear function φ_Z on $\Delta_{Y_\bullet}(D)$. More explicitly, it is given by

$$\begin{aligned} \varphi_Z: \Delta_{Y_\bullet}(D) &\rightarrow \mathbb{R} \\ m &\mapsto \langle m - m_\tau, u_\tau \rangle, \end{aligned}$$

where $m_\tau := m_\sigma$ is part of the Cartier data $\{m_\sigma\}_{\sigma \in \Sigma(n)}$ of D for any cone $\sigma \in \Sigma$ containing τ .

This function φ_Z measures the lattice distance of a given point m in the Newton–Okounkov body to the hyperplane with equation $\langle m, u_\tau \rangle = \langle m_\tau, u_\tau \rangle$. If D is ample this is the lattice distance to a face of $\Delta_{Y_\bullet}(D)$.

Proof. Since the flag Y_\bullet and the divisor D are torus-invariant, the resulting Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ coincides with a translate of the polytope P_D .

Let Y_\bullet^* denote the proper transform of Y_\bullet on X^* . The pullback π_Z^*D of the given divisor D determines a polytope $P_{\pi_Z^*D}$ and by construction we have $P_D \simeq P_{\pi_Z^*D}$. To embed the Newton–Okounkov body $\Delta_{Y_\bullet^*}(\pi_Z^*D) \simeq P_{\pi_Z^*D}$ in \mathbb{R}^n we have to fix a trivialization of the line bundle. Fix the origin $\mathbf{0}$ of \mathbb{R}^n to be m_τ . If $m_\tau \in P_{\pi_Z^*D}$, this means that the corresponding character $\chi^{\mathbf{0}}$ is identified with a global section s of $\mathcal{O}(\pi_Z^*D)$ that does not vanish along Z .

Then according to [CLS11, Proposition 4.1.1] the order of vanishing of a character χ^m along Z is given as

$$\text{ord}_Z(\chi^m) = \text{ord}_E(\chi^m) = \langle m, u_\tau \rangle$$

for $m \in \Delta_{Y_\bullet^*}(\pi_Z^*D)$.

For a given point $m \in \Delta_{Y_\bullet^*}(\pi_Z^*D)$ let $s \in \mathcal{O}(k\pi_Z^*D)$ be an arbitrary global section that gets mapped to m by the flag valuation associated to Y_\bullet^* for some suitable $k \in \mathbb{N}$.

Write s in local coordinates x_i with respect to the flag Y_\bullet^* , that is, Y_i^* is given by $x_1 = \dots = x_i = 0$ and in particular $\mathbf{0} = Y_n^*$. The change of coordinates is obtained by multiplication by the monomial χ^{m_τ} on the level of functions and by a respective translation by the vector $m_\tau \in M$ on the level of points. This yields

$$\text{ord}_Z(\chi^m) = \langle m - m_\tau, u_\tau \rangle.$$

The section s is identified with a linear combination of characters, in which χ^m appears with non-zero coefficient. This gives the upper bound

$$\text{ord}_Z(s) = \min_{m' \in \text{supp}(s)} \text{ord}_Z(\chi^{m'}) \leq \text{ord}_Z(\chi^m).$$

The lower bound is realized by the monomial χ^m itself. Hence, the function that comes from the geometric valuation along the subvariety Z is given as

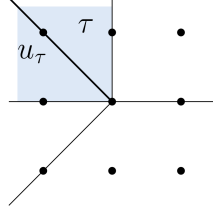
$$\varphi_Z(m) = \text{ord}_Z(\chi^m) = \langle m - m_\tau, u_\tau \rangle$$

for $m \in \Delta_{Y_\bullet^*}(\pi^*D) = \Delta_{Y_\bullet}(D)$. □

We give an example to illustrate the proof.

Example 4.2. As in Example 3.6 we consider the Hirzebruch surface $X = \mathcal{H}_1$, an admissible torus-invariant flag Y_\bullet and the big divisor $D = D_3 + 2D_4$. As a torus-invariant subvariety $Z \subseteq X$ consider the torus fixed point associated to the cone $\tau = \text{cone}((-1, 0), (0, 1))$.

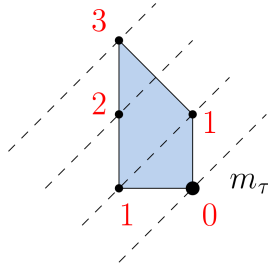
Then the additional primitive ray generator $u_\tau = (-1, 1) = (-1, 0) + (0, 1)$ for the fan Σ^* comes from the star subdivision of the fan Σ relative to the cone τ as indicated

FIGURE 10. Star subdivision of the fan Σ relative to τ

in Figure 10. The Newton–Okounkov function φ_Z on the Newton–Okounkov body $\Delta_{Y_\bullet}(\pi^*D)$ is given by

$$\varphi_Z(m) = \langle m - m_\tau, u_\tau \rangle = \langle m - (1, 0), (-1, 1) \rangle,$$

which gives the values shown in Figure 11.

FIGURE 11. Values of the Newton–Okounkov function φ_Z associated to the m_τ

4.1.1. Interpretation of Subgraph as Newton–Okounkov Body

Let X be a smooth projective variety, Y_\bullet an admissible flag and D a big \mathbb{Q} -Cartier divisor on X . This determines the Newton–Okounkov body $\Delta_{Y_\bullet}(D)$. Given a smooth subvariety $Z \subseteq X$, we consider the function φ_Z on $\Delta_{Y_\bullet}(D)$ that comes from the geometric valuation ord_Z .

In [KMR19] Küronya, Maclean and Roé construct a variety \hat{X} , a flag \hat{Y}_\bullet and a divisor \hat{D} on \hat{X} so that the resulting Newton–Okounkov body is the subgraph of φ_Z over $\Delta_{Y_\bullet}(D)$. We translate their construction into polyhedral language in the toric case.

According to Lemma 4.2 in [KMR19] we may assume that the geometric valuation ord_Z comes from a smooth effective Cartier divisor L on X , i.e. $\text{ord}_Z = \text{ord}_L$. This can always be guaranteed by possibly blowing up X (compare §4.2).

Set

$$\hat{X} := \mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{O}_X(L)).$$

In other words, we consider the total space of the line bundle $\mathcal{O}_X(L)$ and compactify each fiber to a \mathbb{P}^1 . We denote by π the projection $\hat{X} \xrightarrow{\pi} X$. The zero-section of

$\mathcal{O}_X(L)$ is a divisor $X_0 \xrightarrow{\iota_0} \hat{X}$ which is isomorphic to X . The same is true for the ∞ -section $X_\infty \xrightarrow{\iota_\infty} \hat{X}$.

In our toric situation, \hat{X} is again toric, and its fan is described in Proposition 7.3.3 of [CLS11] as follows. The local equation of L as a Cartier divisor along a toric patch U_σ is a torus character which corresponds to a linear function on σ . These linear functions glue to the support function $\text{SF}_L: |\Sigma| \rightarrow \mathbb{R}$ of L (see Definition 4.2.11 & Theorem 4.2.12 in [CLS11]). Using SF_L , we define an upper and a lower cone in $N_{\mathbb{R}} \times \mathbb{R}$ for every $\sigma \in \Sigma$:

$$\begin{aligned} \hat{\sigma} &:= \{(u, h) \in N_{\mathbb{R}} \times \mathbb{R} \mid u \in \sigma, h \geq \text{SF}_L(u)\} \\ \check{\sigma} &:= \{(u, h) \in N_{\mathbb{R}} \times \mathbb{R} \mid u \in \sigma, h \leq \text{SF}_L(u)\}. \end{aligned}$$

Together with their faces, these cones form a fan $\hat{\Sigma}$ which determines our \hat{X} . The upper and lower cones of the origin $\mathbf{0} \in \Sigma$, are rays $\hat{\mathbf{0}}$ and $\check{\mathbf{0}}$ whose toric divisors in \hat{X} are X_0 and X_∞ , respectively. The projection $\hat{X} \rightarrow X$ is toric. It comes from the projection $N \times \mathbb{Z} \rightarrow N$ which identifies both $\text{star}(\hat{\mathbf{0}})$ and $\text{star}(\check{\mathbf{0}})$ with Σ .

Example 4.3. Let $X_\Sigma = \mathbb{P}^1$ be the projective line. Its corresponding fan Σ in \mathbb{R} is depicted in Figure 12, where the torus-invariant prime divisors D_0 and D_1 correspond to $\sigma_0 = \mathbb{R}_{\geq 0}$ and $\sigma_1 = \mathbb{R}_{\leq 0}$ with primitive ray generators $u_0 = 1$ and $u_1 = -1$, respectively.

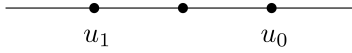


FIGURE 12. The fan Σ of the projective line $X_\Sigma = \mathbb{P}^1$

Consider the divisor $L = D_0$. Then $\hat{\Sigma}$ is a fan in \mathbb{R}^2 and its top-dimensional cones are

$$\begin{aligned} \hat{\sigma}_0 &= \text{cone}((0, 1), (1, -1)), \\ \check{\sigma}_0 &= \text{cone}((0, -1), (1, -1)), \\ \hat{\sigma}_1 &= \text{cone}((0, 1), (-1, 0)) \text{ and} \\ \check{\sigma}_1 &= \text{cone}((0, -1), (-1, 0)). \end{aligned}$$

We obtain the fan of the Hirzebruch surface \mathcal{H}_1 as depicted in Figure 13.

For the suitable divisor \hat{D} on \hat{X} we fix some rational number b such that

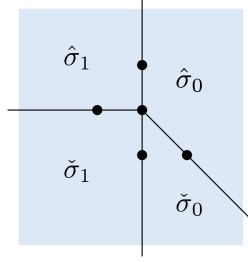
$$b > \sup\{t > 0 \mid D - tL \text{ is big}\}$$

and define $\hat{D} := \pi^*D + bX_\infty$. As an admissible flag \hat{Y}_\bullet we set

$$\hat{Y}_1 := X_0, \hat{Y}_i := \iota_0(Y_{i-1}) \text{ for all } i \geq 2.$$

Küronya, Maclean and Roé show that \hat{X} , \hat{Y}_\bullet and \hat{D} are the suitable objects to obtain the desired identification

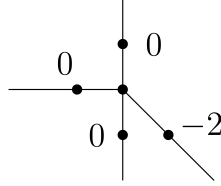
$$\{(m, h) \in \Delta_{Y_\bullet}(D) \times \mathbb{R} \mid 0 \leq h \leq \varphi_L(m)\} = \Delta_{\hat{Y}_\bullet}(\hat{D}).$$

FIGURE 13. The fan $\hat{\Sigma}$ of $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$

Example 4.4. We continue with Example 4.3. In addition to the data $X = \mathbb{P}^1$, $L = D_0$ we choose the toric flag $Y_1 = V(\sigma_0)$ and the big divisor $D = 2D_0$.

Then the flag \hat{Y}_\bullet consists of $\hat{Y}_1 = V(\hat{\mathbf{0}})$ and $\hat{Y}_2 = \iota_0(Y_1) = V(\hat{\sigma}_0)$.

The support function for π^*D is the pullback of the support function for D along the linear projection $\hat{\Sigma} \rightarrow \Sigma$. Its values at the ray generators are indicated in Figure 14.

FIGURE 14. Values of the support function SF_{π^*D} for the ray generators u_ρ for $\rho \in \hat{\Sigma}(1)$

The resulting polyhedron in $M_{\mathbb{R}} \times \mathbb{R}$ is

$$\{(m, h) \in M_{\mathbb{R}} \times \mathbb{R} \mid m \in \Delta_{Y_\bullet}(D), h \leq 0, h \geq 0\}$$

Adding $tX_\infty = tV(\hat{\mathbf{0}})$ to π^*D for $t > 0$, relaxes the corresponding inequality $h \leq 0$ to $h + t \leq 0$. The effect on the polyhedron is depicted in Figure 15.

4.2. Geometric Valuation Coming from a General Point. Let X now be a smooth projective toric surface and D an ample divisor on X . In this section we relax the requirements in the sense that the function φ_R now comes from the geometric valuation ord_R at a general point R , not necessarily torus-invariant. Here we can determine the values of φ_R on parts of $\Delta_{Y_\bullet}(D)$ and give an upper bound on the entire Newton–Okounkov body.

Therefore we need to introduce some more terminology. We are given an admissible torus-invariant flag $Y_\bullet: X \supseteq Y_1 \supseteq Y_2$ on X . Since Y_\bullet is toric, the Newton–Okounkov body $\Delta_{Y_\bullet}(D) \subseteq \mathbb{R}^2$ is isomorphic to P_D and one of its facets corresponds to Y_1 . Let $u \in (\mathbb{R}^2)^*$ denote the defining linear functional that selects this face when minimized over the polytope $\Delta_{Y_\bullet}(D)$. We denote by $F \preceq \Delta_{Y_\bullet}(D)$ the face that is selected, when

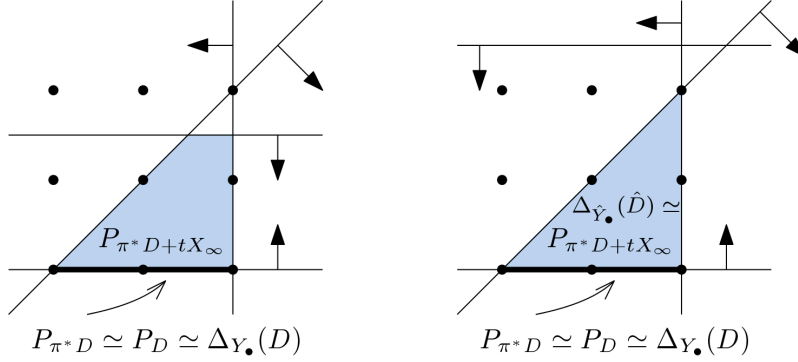


FIGURE 15. The polytope $P_{\pi^*D+tX_\infty}$ for small t on the left and for $t = b$ on the right

maximizing u over $\Delta_{Y_\bullet}(D)$. Either this already is a vertex or if not, we maximize u' over F , where $u' \in (\mathbb{R}^2)^*$ is a linear functional selecting Y_2 when minimized over $\Delta_{Y_\bullet}(D)$. Denote the resulting vertex by Q_{Y_\bullet} . We say that the vertex Q_{Y_\bullet} lies at the *opposite side* of the polytope P_D with respect to the flag Y_\bullet .

Proposition 4.5. *Let X be a smooth projective toric surface, D an ample divisor and Y_\bullet an admissible torus-invariant flag on X . Denote by $\Delta_{Y_\bullet}(D)$ the corresponding Newton–Okounkov body and by $Q = Q_{Y_\bullet}$ the vertex at the opposite side of $\Delta_{Y_\bullet}(D)$ with respect to Y_\bullet . Moreover let $R \in \mathbb{T}$ be a general point. Then for the Newton–Okounkov function φ_R coming from the geometric valuation ord_R we have*

(1)

$$\varphi_R(a, b) \leq a + b$$

for all $(a, b) \in \Delta_{Y_\bullet}(D)$, where (a, b) are the coordinates in the coordinate system associated to Q .

(2) Furthermore, we have

$$\varphi_R(a, b) = a + b$$

for all

$$(a, b) \in \{(a', b') \in \Delta_{Y_\bullet}(D) \mid \text{NP}((x-1)^{a'}(y-1)^{b'}) \subseteq \Delta_{Y_\bullet}(D)\}.$$

Proof. (1) Let $(a, b) \in \Delta_{Y_\bullet}(D)$ be a valuative point in the Newton–Okounkov body. We want to determine $\varphi_R(a, b)$, where φ_R is the function coming from the geometric valuation ord_R . Consider an arbitrary section $s \in \Gamma(X, \mathcal{O}_X(kD))$ that is mapped to $(a, b) = \frac{1}{k} \text{val}_{Y_\bullet}(s)$ for some $k \in \mathbb{N}$. Then by construction the rescaled exponent vectors of all monomials that can occur in s have to be an element of the set

$$H^+ := \{m \in \Delta_{Y_\bullet}(D) \mid u(m) > u(a, b) \text{ or } (u(m) = u(a, b) \text{ and } u'(m) > u'(a, b))\}.$$

As indicated in Figure 16 this region is obtained by intersecting $\Delta_{Y_\bullet}(D)$ with the positive halfspace associated to the hyperplane $H = \{m \mid u(m) = u(a, b)\}$.

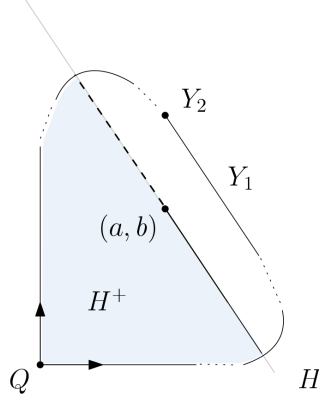


FIGURE 16. Admissible region H^+ of rescaled exponent vectors associated to monomials of s inside the Newton–Okounkov body $\Delta_{Y_\bullet}(D)$

Moreover, we can assume without loss of generality that the general point R is given as $R = (1, 1)$. To determine the order of vanishing of s at R we substitute x by $x' + 1$ and y by $y' + 1$ and bound the order of vanishing of $s'(x', y') = s(x' + 1, y' + 1)$ at $(0, 0)$. Assuming without loss of generality that the monomial $x^{ka}y^{kb}$ itself occurs in s with coefficient 1, multiplying out gives

$$\begin{aligned} s'(x', y') &= s(x' + 1, y' + 1) = (x' + 1)^{ka}(y' + 1)^{kb} + \dots \\ &= (x')^{ka}(y')^{kb} + \text{lower order terms} + \dots \end{aligned}$$

CLAIM: The monomial $(x')^{ka}(y')^{kb}$ cannot be canceled out by terms coming from \dots .

Aiming at a contradiction assume that \dots contains a monomial $(x' + 1)^{kc}(y' + 1)^{kd}$ for some $kc, kd \in \mathbb{N}$ that produces $(x')^{ka}(y')^{kb}$ when multiplied out. Observe that multiplying out $(x' + 1)^{kc}(y' + 1)^{kd}$ produces all monomials in $\{(x')^e(y')^f \mid e \leq kc \text{ and } f \leq kd\}$. Thus $kc \geq ka$ and $kd \geq kb$. In addition, as an exponent vector of a monomial in s the point (c, d) is required to be an element of the set H^+ , which forces the hyperplane H to have positive slope as indicated in Figure 17.

Let $F' \leq \Delta_{Y_\bullet}(D)$ denote the face that corresponds to $\{x = 0\}$ and let Q' denote its second vertex. Then $u(Q') > u(Q)$ which contradicts the fact that u is maximized at Q over $\Delta_{Y_\bullet}(D)$. Thus such a monomial $(x' + 1)^{kc}(y' + 1)^{kd}$ cannot exist and $(x')^{ka}(y')^{kb}$ does not cancel out.

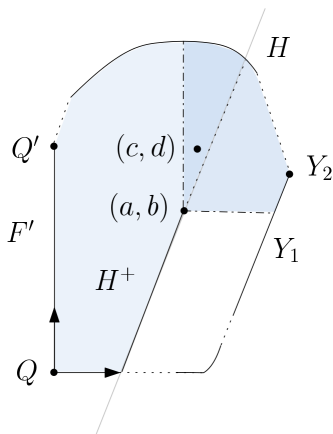


FIGURE 17. A non-empty region of points $(c, d) \in H^+$ that satisfy $c \geq a$ and $d \geq b$ forcing H to have positive slope

Consequently, $k(a + b)$ is an upper bound for the order of vanishing of s' at $(0, 0)$ and thus for s at R . Since this is true for all sections s that get mapped to (a, b) , this yields $\varphi_R(a, b) \leq a + b$.

(2) Consider a point

$$(a, b) \in \text{Par} := \{(a', b') \mid \text{NP}((x - 1)^{a'}(y - 1)^{b'}) \subseteq \Delta_{Y_\bullet}(D)\} ,$$

and set $s(x, y) = (x - 1)^{ka}(y - 1)^{kb}$, for a $k \in \mathbb{N}$ such that s is a global section of kD , whose Newton polytope can be seen in Figure 18. Then by construction s is a section associated to the point (a, b) and its Newton polytope fits inside $k\Delta_{Y_\bullet}(D)$. We have $\text{ord}_R(s) = k(a + b)$ which gives the lower bound $\varphi_R(a, b) \geq \frac{1}{k} \text{ord}_R(s)$. Combined with (1) we obtain $\varphi_R(a, b) = a + b$.

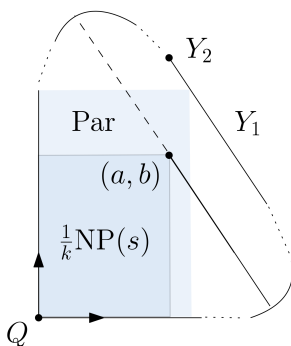


FIGURE 18. The scaled Newton polytope $\frac{1}{k} \text{NP}(s)$ of the section $s(x, y) = (x - 1)^{ka}(y - 1)^{kb}$

□

We illustrate the use of Proposition 4.5 by the following example.

Example 4.6. We continue our running example of the Hirzebruch surface $X = \mathcal{H}_1$ and the ample divisor $D = D_3 + 2D_4$ as in Example 3.6. Furthermore fix the torus-invariant flag $Y_\bullet: X \supseteq Y_1 \supseteq Y_2$, where $Y_1 = D_1$ and $Y_2 = D_1 \cap D_2$. Then the vertex $Q = Q_{Y_\bullet}$ of the Newton–Okounkov body $\Delta_{Y_\bullet}(D) \simeq P_D$ that lies at the opposite side of the polytope P_D with respect to the flag Y_\bullet is the one indicated in Figure 19. The associated coordinate system specifies coordinates a, b for the plane \mathbb{R}^2 and local toric coordinates x, y .

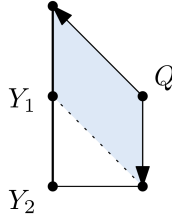


FIGURE 19. The coordinate system associated to the vertex Q which lies at the opposite side of P_D with respect to the flag Y_\bullet .

We want to determine the Newton–Okounkov function coming from the geometric valuation at the point $R = (1, 1)$. Proposition 4.5 yields the upper bound $\varphi_R(a, b) \leq a + b$ on the entire Newton–Okounkov body $\Delta_{Y_\bullet}(D)$, and $\varphi_R(a, b) = a + b$ for (a, b) satisfying $a, b \leq 1$, as indicated by the shaded region in Figure 19. It will turn out in Example 4.8 that there exist points $(a, b) \in \Delta_{Y_\bullet}(D)$ for which we have $\varphi_R(a, b) < a + b$.

For a particularly nice class of polygons, Proposition 4.5 alone is enough to determine the function φ_R . A polytope $P \subseteq \mathbb{R}_{\geq 0}^n$ is called anti-blocking if $P = (P + \mathbb{R}_{\leq 0}^n) \cap \mathbb{R}_{\geq 0}^n$ (compare [Ful71, Ful72]). Observe that this coordinate dependent property implies (and for $n = 2$ is equivalent to) the fact that the parallelepiped spanned by the edges at the origin covers P .

Corollary 4.7. *Let X, Y_\bullet, D and R be as in Proposition 4.5. Suppose $\Delta_{Y_\bullet}(D)$ is anti-blocking. Let Y'_\bullet be a torus-invariant flag opposite to the origin. Then the Newton–Okounkov function φ_R on $\Delta_{Y'_\bullet}(D)$ is given by*

$$\varphi_R(a, b) = a + b$$

on the entire Newton–Okounkov body $\Delta_{Y'_\bullet}(D) \simeq P_D$ in the coordinate system associated to Y'_\bullet .

Using the tools from Section 5, Corollary 4.7 implies that the Seshadri constant of D at R is rational. This can also be seen from Sano’s Theorem [San14] as $\dim | -K_X | \geq 3$ in the anti-blocking case.

If we are not in the lucky situation of Corollary 4.7, then things are getting more complicated and more interesting. We give an approach that works in numerous cases. We consider this strategy as the main contribution of the article.

The general strategy

GIVEN:

- X a smooth projective toric surface
- Y_\bullet an admissible torus-invariant flag
- D an ample torus-invariant divisor on X
- R a general point on \mathbb{T} .

GOAL: Determine the function

$$\varphi_R: \Delta_{Y_\bullet}(D) \simeq P_D \rightarrow \mathbb{R}.$$

coming from the geometric valuation ord_R .

APPROACH:

- (1) For each valutive point $(a, b) \in \Delta_{Y_\bullet}(D)$ “guess” a Newton polytope $\frac{1}{k} \text{NP}(s) \subseteq P_D$ of a global section $s \in \Gamma(X, \mathcal{O}_X(kD))$ for some $k \in \mathbb{N}$ to maximize the order of vanishing $\text{ord}_R(s)$ according to the following rules:
 - The section s has to correspond to the point (a, b) .
 - Choose a Newton polytope $\text{NP}(s)$ that is a zonotope whose edge directions all come from edges in P_D .
 - Try to maximize the perimeter of the Newton polytope $\text{NP}(s)$ among the above.
- (2) Determine the values of the function $\varphi: \Delta_{Y_\bullet}(D) \rightarrow \mathbb{R}$ that takes $\frac{1}{k} \text{ord}_R(s)$ as a value with respect to the chosen sections s for a point $(a, b) \in \Delta_{Y_\bullet}(D)$ and compute the integral $\int_{\Delta_{Y_\bullet}(D)} \varphi$.
- (3) Choose a new admissible flag $Y'_\bullet: X \supseteq Y'_1 \supseteq Y'_2$ such that
 - $Y'_1 = \overline{\{x^v - 1\}}$ is a curve on X given by a binomial for some primitive $v \in \mathbb{Z}^2$ with $R \in Y'_1$. Hereby choose the vector v in such a way that the width $\text{width}_u(P_D)$ is minimized, where $u \in (\mathbb{R}^2)^*$ is a linear functional with $u(v) = 0$.
 - $Y'_2 = R$.
- (4) Compute the Newton–Okounkov body $\Delta_{Y'_\bullet}(D)$ with respect to that new flag Y'_\bullet using variation of Zariski decomposition or the combinatorial methods from Section 3.

- (5) Compute the integral $\int_{\Delta_{Y'_\bullet}(D)} \varphi'$, where we assume the function to be given by

$$\begin{aligned} \varphi' : \Delta_{Y'_\bullet}(D) &\rightarrow \mathbb{R} \\ (a', b') &\mapsto a' + b'. \end{aligned}$$

- (6) Compare the value of the integrals $\int_{\Delta_{Y_\bullet}(D)} \varphi$ and $\int_{\Delta_{Y'_\bullet}(D)} \varphi'$. It holds that

$$(16) \quad \int_{\Delta_{Y_\bullet}(D)} \varphi \leq \int_{\Delta_{Y_\bullet}(D)} \varphi_R = \int_{\Delta_{Y'_\bullet}(D)} \varphi'_R \leq \int_{\Delta_{Y'_\bullet}(D)} \varphi'.$$

- If $\int_{\Delta_{Y_\bullet}(D)} \varphi = \int_{\Delta_{Y'_\bullet}(D)} \varphi'$ then we have equality in (16) and therefore a certificate, that the choices we have made were valid and we are done.
- If $\int_{\Delta_{Y_\bullet}(D)} \varphi < \int_{\Delta_{Y'_\bullet}(D)} \varphi'$, then we have chosen sections with non-maximal orders of vanishing at R in step (1) or for the chosen vector v from step (3) the function φ_R takes values smaller than $a' + b'$ somewhere on $\Delta_{Y'_\bullet}(D)$.

Example 4.8. We return to Example 4.6, and again consider the Hirzebruch surface $X = \mathcal{H}_1$ equipped with the torus-invariant flag $Y_\bullet : X \supseteq Y_1 \supseteq Y_2$, where $Y_1 = D_1$ and $Y_2 = D_1 \cap D_2$ and the ample divisor $D = D_3 + 2D_4$ on X .

We want to determine the values of a function on $\Delta_{Y_\bullet}(D)$ coming from a geometric valuation at a general point, so let $R = (1, 1) \in X$ in local coordinates. The valuation ord_R is given by the order of vanishing of a section at R . The value of the function φ_R associated to ord_R at a point of $\Delta_{Y_\bullet}(D)$ is defined as the supremum of orders of vanishing at R over all sections that get mapped to this point. More precisely for the rational points in $\Delta_{Y_\bullet}(D)$ in the coordinate system associated to the flag Y_\bullet we study

$$\begin{aligned} \varphi_R : \Delta_{Y_\bullet}(D) &\rightarrow \mathbb{R} \\ (a, b) &\mapsto \lim_{k \rightarrow \infty} \frac{1}{k} \sup \{t \in \mathbb{R} \mid \text{there exists } s \in \mathcal{O}_X(kD) : \\ &\quad \nu_{Y_\bullet}(s) = k(a, b), \text{ord}_R(s) \geq t\} . \\ &= \lim_{k \rightarrow \infty} \frac{1}{k} \sup \{t \in \mathbb{R} \mid \text{there exists } s \in \mathcal{O}_X(kD) : \\ &\quad \text{ord}_{Y_1}(s) = ka, \text{ord}_{Y_2}(s) = kb, \text{ord}_R(s) \geq t\} , \end{aligned}$$

where s_1 is given as in (3).

We claim that φ_R coincides with the function φ given by

$$\varphi(a, b) = \begin{cases} 2 - a & \text{if } 0 \leq a + b \leq 1 \\ 3 - 2a - b & \text{if } 1 \leq a + b \leq 2 \end{cases}$$

at a point $(a, b) \in \Delta_{Y_\bullet}(D)$.

To verify this claim we will give explicit respective sections and argue that the maximal value of ord_R is achieved for these particular sections. We treat the two cases individually.

$$\boxed{0 \leq a + b \leq 1:}$$

Set

$$s(x, y) = (x^a(y - 1)^{2-a-b}(x - y)^b)^k$$

in local coordinates x, y for suitable $k \in \mathbb{N}$. The corresponding Newton polytope $\frac{1}{k} \text{NP}(s)$ is depicted in Figure 20. Since the leftmost part of it has coordinates (a, \cdot) , we have $\text{ord}_{Y_1}(s) = ka$. If we restrict to the line (a, \cdot) , then the lowest point of the Newton polytope is (a, b) and thus $\text{ord}_{Y_2}(s_1) = kb$. Together with the fact that the Newton polytope $\frac{1}{k} \text{NP}(s)$ fits inside the Newton-Okounkov body $\Delta_{Y_\bullet}(D)$ this guarantees that the section s is actually mapped to the point (a, b) when computing $\Delta_{Y_\bullet}(D)$.

For the order of vanishing of interest we obtain

$$\text{ord}_R(s) = k((2 - a - b) + b) = k(2 - a) .$$

$$\boxed{1 \leq a + b \leq 2:}$$

Set

$$s(x, y) = (x^a y^{a+b-1} (y - 1)^{2-a-b} (x - y)^{1-a})^k .$$

That all the requirements are fulfilled by s follows by using the same arguments as in the previous case. For the order of vanishing of interest we obtain

$$\text{ord}_R(s) = k((2 - a - b) + (1 - a)) = k(3 - 2a - b) .$$

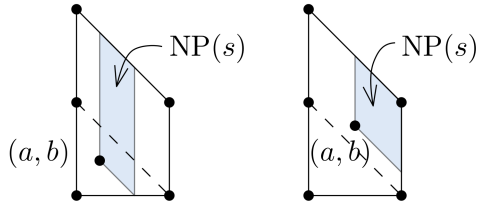


FIGURE 20. Newton polytopes $\text{NP}(s)$ of the respective sections s

The values of the resulting piecewise linear function are depicted in Figure 21.

If we integrate φ over $\Delta_{Y_\bullet}(D)$ we obtain

$$\int_{\Delta_{Y_\bullet}(D)} \varphi = \frac{11}{6} .$$

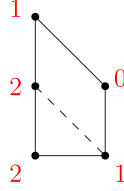


FIGURE 21

Now it remains to show that these values are actually the maximal ones that can be realized. To do this we make use of the fact that the integral of our function φ over the Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ is independent of the flag Y_\bullet .

We keep the underlying variety X the ample divisor D . Choose a new admissible flag $Y'_\bullet: X \supseteq Y'_1 \supseteq Y'_2$, where Y'_1 is the curve defined by the local equation $y - 1 = 0$ and $Y'_2 = R = (1, 1)$ is the point of the geometric valuation. Since this flag is no longer torus-invariant, the corresponding Newton–Okounkov body $\Delta_{Y'_\bullet}(D)$ will differ from the polytope P_D . As shown in Example 3.6 we obtain the new Newton–Okounkov body $\Delta_{Y'_\bullet}(D)$ depicted in Figure 5.

For the function φ'_R on $\Delta_{Y'_\bullet}(D)$ we are still working with the geometric valuation associated to ord_R . Thus define

$$\begin{aligned} \varphi': \Delta_{Y'_\bullet}(D) &\rightarrow \mathbb{R} \\ (a', b') &\mapsto a' + b'. \end{aligned}$$

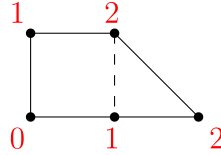


FIGURE 22

The values of φ' are depicted in Figure 22. If we integrate φ' over $\Delta_{Y'_\bullet}(D)$ we obtain

$$\int_{\Delta_{Y'_\bullet}(D)} \varphi' = \frac{11}{6}.$$

Overall we have $\int_{\Delta_{Y_\bullet}(D)} \varphi = \int_{\Delta_{Y'_\bullet}(D)} \varphi'$. This shows that our choice for the section s was indeed maximal with respect to $\text{ord}_R(s)$ and thus determines the value of φ_R .

Remark 4.9. In the previous example the integrals $\int_{\Delta_{Y_\bullet}(D)} \varphi$ and $\int_{\Delta_{Y'_\bullet}(D)} \varphi'$ coincide. Observe that even more is true. Let

$$G(\varphi) = \{(a, b, \varphi(a, b)) \mid (a, b) \in \Delta_{Y_\bullet}(D)\}$$

denote the graph of φ . Since φ is a concave and piecewise linear function, the set

$$\Delta_{Y_\bullet}(D)_\varphi := \text{conv}((\Delta_{Y_\bullet}(D) \times \{0\}) \cup G(\varphi)) \subseteq \mathbb{R}^3$$

is a 3-dimensional polytope. If we compare $\Delta_{Y_\bullet}(D)_\varphi$ and $\Delta_{Y'_\bullet}(D)_{\varphi'}$ it turns out that they are $\text{SL}_3(\mathbb{Z})$ -equidecomposable, where the respective maps are volume preserving.

To see this we give the explicit maps between corresponding pieces. Use

$$\begin{aligned} \psi_1: \mathbb{R}^3 &\rightarrow \mathbb{R}^3 \\ \begin{pmatrix} a \\ b \\ c \end{pmatrix} &\mapsto \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

to map the parallelogram in $\Delta_{Y_\bullet}(D)$ with its corresponding heights. And use

$$\begin{aligned} \psi_2: \mathbb{R}^3 &\rightarrow \mathbb{R}^3 \\ \begin{pmatrix} a \\ b \\ c \end{pmatrix} &\mapsto \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

for mapping the triangle in $\Delta_{Y_\bullet}(D)$ with its corresponding heights. This is illustrated in Figure 23, where the respective heights are given in red.

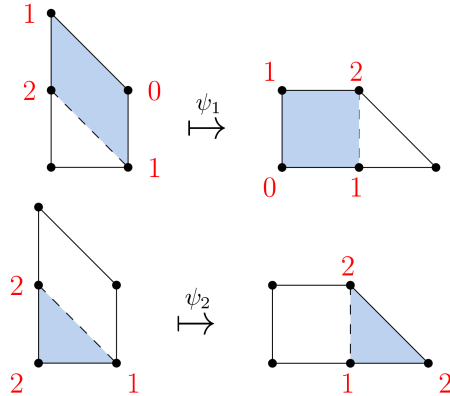


FIGURE 23. $\text{SL}_3(\mathbb{Z})$ -equidecomposable pieces of the Newton-Okounkov bodies $\Delta_{Y_\bullet}(D)$ on the left and $\Delta_{Y'_\bullet}(D)$ on the right

We conjecture that this is not a coincidence but holds in general for a suitable choice of the vector v .

Conjecture 4.10. Let X be a smooth projective toric surface, D an ample torus-invariant divisor, $R = (1, 1)$ a general point and $v \in \mathbb{Z}^2$ a primitive direction. Any admissible torus-invariant flag Y_\bullet gives the Newton–Okounkov body $\Delta_{Y_\bullet}(D) \simeq P_D$. Denote by $\Delta_{Y'_\bullet}(D)$ the Newton–Okounkov body with respect to the flag Y'_\bullet : $X \supseteq C \supseteq \{R\}$, where $C = \overline{\{x^v - 1\}}$ and $R = (1, 1)$. If the Newton–Okounkov function φ'_R coming from the geometric valuation ord_R is given by $\varphi'_R(a', b') = a' + b'$ for $(a', b') \in \Delta_{Y'_\bullet}(D)$, then for the flag Y_\bullet at the opposite side of $Q = \Psi^{-1}(0, 0)$ we have $\varphi_R(a, b) = \varphi'_R(\Psi(a, b))$ for all $(a, b) \in \Delta_{Y_\bullet}(D)$. Here Ψ is the piecewise linear isomorphism from Section 3 between the Newton–Okounkov bodies.

5. RATIONALITY OF CERTAIN SESHADRI CONSTANTS ON TORIC SURFACES

There is a direct link between the rationality of Seshadri constants on surfaces and the integral of Newton–Okounkov functions. Let X be a smooth projective surface, D an ample divisor and $z \in X$ a point. We denote the blow-up of z with exceptional divisor E by $\pi: X' \rightarrow X$. The Seshadri constant is the invariant

$$(17) \quad \varepsilon(X, D; z) := \sup\{t > 0 \mid \pi^*D - tE \text{ is nef}\}.$$

Rationality can be deduced from the rationality of the associated integral in the following way.

Corollary 5.1 ([KMR19], Corollary 4.5). *Let X be a smooth projective surface, $z \in X$ and D an ample Cartier divisor on X . Then $\varepsilon(X, D; z)$ is rational if $\int_{\Delta_{Y_\bullet}(D)} \varphi_z$ is rational, where φ_z is the Newton–Okounkov function coming from the geometric valuation associated to z and Y_\bullet any admissible flag.*

In [San14] Sano studies Seshadri constants on rational surfaces with anticanonical pencils. More precisely he considers a smooth rational surface X that is either a composition of blow-ups of \mathbb{P}^2 or of a Hirzebruch surface \mathcal{H}_d , such that $\dim | -K_X | \geq 1$. In terms of the corresponding polytope this means that P_{-K_X} contains at least two lattice points. In these cases he gives explicit formulas for the Seshadri constant $\varepsilon(X, \mathcal{L}; z)$ of an ample line bundle \mathcal{L} at a general point $z \in X$ (Theorem 3.3 and Corollary 4.12 in [San14]). As a consequence he obtains rationality in the cases above as observed in Remark 4.2.

In [Lun20] Lundman computes Seshadri constants at a general point for some classes of smooth projective toric surfaces. It follows in particular that it is rational in these cases. The characterization of the classes involve the following definitions.

Definition 5.2. *Let \mathcal{L} be a line bundle on a smooth variety X and $z \in X$ a smooth point with maximal ideal $\mathfrak{m}_z \subseteq \mathcal{O}_X$. Consider the map*

$$j_z^k: \Gamma(X, \mathcal{L}) \rightarrow \Gamma(X, \mathcal{L} \otimes \mathcal{O}_X/\mathfrak{m}_z^{k+1})$$

$$s \mapsto \left(s(z), \dots, \frac{\partial^t s}{\partial z^t}(z), \dots \right)_{t \leq k},$$

where $\underline{z} = (z_1, \dots, z_n)$ is a local system of coordinates around z . We say that \mathcal{L} is k -jet spanned at z if the map j_z^k is surjective. We denote by $\mathfrak{s}(\mathcal{L}, z)$ the largest k such that X is k -jet spanned at z and call it the degree of jet separation of \mathcal{L} at z .

So the map j_z^k takes s to the terms of degree at most k in the Taylor expansion of s around z . For X a projective toric variety let s_0, \dots, s_d be a basis for $\Gamma(X, \mathcal{L})$. Then \mathcal{L} is k -jet spanned at $z \in X$ if and only if the matrix of k -jets

$$J_k(\mathcal{L}) := (J_k(\mathcal{L}))_{i,j} := \left(\frac{\partial^{|v|}}{\partial_{z_{v_1}} \partial_{z_{v_2}} \dots \partial_{z_{v_n}}} (s_i) \right)_{0 \leq i \leq d, 0 \leq |v| \leq k}$$

has maximal rank when evaluated at z , where $v = (v_1, \dots, v_n) \in \mathbb{N}^n$ and $|v| = |v_1 + \dots + v_n|$.

Definition 5.3. Let X be a smooth projective toric variety and D a torus-invariant divisor on X . We define

$$\eta(D) := (\sup\{r > 0 \mid P_{rK_X + D} \text{ is non-empty}\})^{-1}$$

and call the polytope $\text{core}(P_D) := P_{\eta(D)^{-1}K_X + D}$ the core of P_D .

Theorem 5.4 ([Lun20], Theorem 1). Let X be a smooth toric surface and \mathcal{L} an ample line bundle. If X is a projective bundle or $\mathfrak{s}(\mathcal{L}, 1) \leq 2$, then $\varepsilon(X, \mathcal{L}; 1) = \mathfrak{s}(\mathcal{L}, 1)$.

The other Theorems in [Lun20] that yield rationality of Seshadri constants both require the core $\text{core}(P_D)$ to be a line segment.

We give an example for which none of the above Theorems applies and thus for which the rationality of the Seshadri constant has not been known before.

Example 5.5. We consider a blow-up $\pi: X \rightarrow \mathbb{P}^2$ of the projective plane in 13 points, namely the toric variety X whose associated fan Σ is depicted in Figure 24.

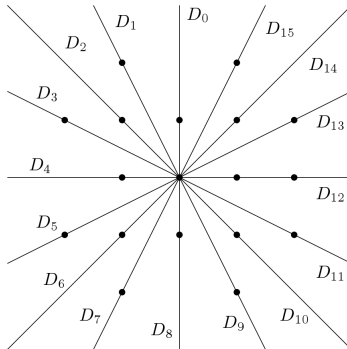


FIGURE 24. The fan Σ with associated torus-invariant prime divisors D_0, \dots, D_{15}

The torus-invariant prime divisors are denoted by D_0, \dots, D_{15} and choose $D = D_1 + 2D_2 + 6D_3 + 5D_4 + 15D_5 + 11D_6 + 19D_7 + 9D_8 + 18D_9 + 10D_{10} + 13D_{11} + 4D_{12} + 4D_{13} + D_{14}$ as an ample divisor on X . For any torus-invariant flag Y_\bullet this gives the polytope P_D in Figure 25 as the Newton–Okounkov body $\Delta_{Y_\bullet}(D)$. We have $\dim | -K_X| = 0$, the core $\text{core}(P_D)$ is a point and the degree of jet separation is $\mathfrak{s}(\mathcal{L}, 1) = 9$. Thus this example does not fall in any of the classes covered by Sano or Lundman.

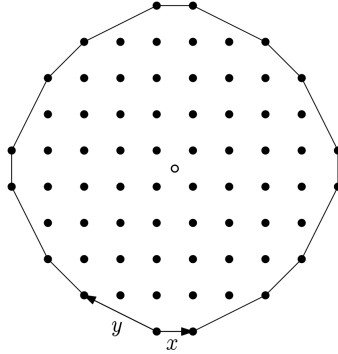


FIGURE 25. The Newton–Okounkov body $\Delta_{Y_\bullet}(D) \simeq P_D$ with $\text{core}(P_D)$

We claim that the Seshadri constant $\varepsilon(X, D; 1)$ is rational. To verify this claim we consider the Newton–Okounkov function φ'_R on $\Delta_{Y'_\bullet}(D)$ coming from the geometric valuation ord_R at the general point $R = (1, 1)$ and argue that its integral takes a rational value. In order to do this consider the flag $Y'_\bullet: X \supseteq Y'_1 \supseteq Y'_2$, where Y'_1 is the curve given by the local equation $x^{-1} - 1 = 0$ and $Y'_2 = R$. Thus we obtain the Newton–Okounkov body $\Delta_{Y'_\bullet}(D)$ with respect to this flag by the shifting process via the vector $v = (-1, 0)$ as explained in Section 3. This gives the polytope $\Delta_{Y'_\bullet}(D)$ shown in Figure 26.

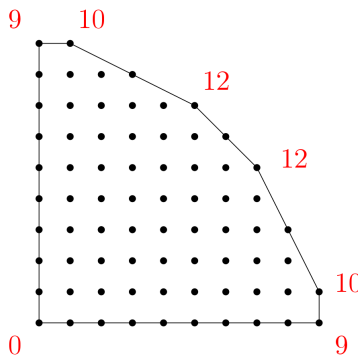


FIGURE 26. The Newton–Okounkov body $\Delta_{Y'_\bullet}(D)$ with respective values of φ'_R

We claim that the Newton-Okounkov function φ'_R on $\Delta_{Y'_\bullet}(D)$ that comes from the geometric valuation ord_R is given as $\varphi'_R(a', b') = a' + b'$ for all $(a', b') \in \Delta_{Y'_\bullet}(D)$. To prove this we consider the following global sections of $\Gamma(X, \mathcal{O}_X(D))$ as in Table 1.

global section s	image in $\Delta_{Y_\bullet}(D)$	image in $\Delta_{Y'_\bullet}(D)$	$\text{ord}_R(s)$
$s_1(x, y) = (x - 1)(x^2y - 1)^9$	(0, 0)	(1, 9)	10
$s_2(x, y) = (x^2y - 1)^9$	(1, 0)	(0, 9)	9
$s_3(x, y) = x^4y^4(x - 1)^9(x^2y - 1)$	(4, 4)	(9, 1)	10
$s_4(x, y) = x^6y^5(x - 1)^9$	(6, 5)	(9, 0)	9
$s_5(x, y) = y(x - 1)^5(x^2y - 1)^7$	(0, 1)	(5, 7)	12
$s_6(x, y) = xy^2(x - 1)^7(x^2y - 1)^5$	(1, 2)	(7, 5)	12
$s_7(x, y) = x^{19}y^9$	(19, 9)	(0, 0)	0

TABLE 1. Global sections of D that realize lower bounds for the order of vanishing ord_R .

The sections are chosen in a way such that they get mapped to the vertices when building the new Newton-Okounkov body $\Delta_{Y'_\bullet}(D)$ and a such that the order of vanishing is $\text{ord}_R(s) = a' + b'$ for a section s that gets mapped to the point $(a', b') \in \Delta_{Y'_\bullet}(D)$. For the vertices $\text{vert}(\Delta_{Y'_\bullet}(D))$ these values realize a lower bound for the function φ'_R . Since the function φ'_R has to be concave, this yields $\varphi'_R(a', b') = a' + b'$ on the entire Newton-Okounkov body. For the integral we obtain

$$\int_{\Delta_{Y'_\bullet}(D)} \varphi'_R = \frac{1295}{3},$$

which is rational and therefore the Seshadri constant $\varepsilon(X, D; 1)$ is rational.

Although proving rationality of the Seshadri constant did not require knowing the values of the function φ_R on the Newton-Okounkov body $\Delta_{Y_\bullet}(D)$, determining these values in this particular example is of independent interest. It turns out that the approach of choosing sections whose Newton polytopes are zonotopes with prescribed edge directions is not always sufficient to maximize the order of vanishing at the general point R . For the function φ_R we expect 22 domains of linearity as shown in Figure 27 that arise from the shifting process in the direction of $v = (-1, 0)$.

As seen in Table 2 for the domains 1, \dots , 9, 13, 14, 15 and 19 zonotopes using only edge directions of $\Delta_{Y_\bullet}(D)$ are sufficient. For the domains 10, 11 and 12 we need a Minkowski sum of those edge directions and “small” triangles that have a high order of vanishing at $(1, 1)$. The section $s(x, y) = x^3y^2 - 3xy + y + 1$ for instance has order of vanishing $\text{ord}_R(s) = 2$ and its Newton polytope $\text{NP}(s)$ is depicted in Figure 28.

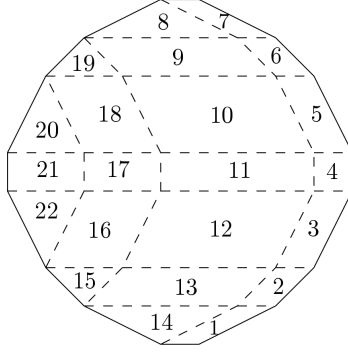


FIGURE 27. Expected domains of linearity of the function φ_R on the Newton–Okounkov body $\Delta_{Y_\bullet}(D)$

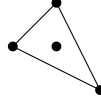


FIGURE 28. The Newton polytope $\text{NP}(s)$ of the section $s(x, y) = x^3y^2 - 3xy + y + 1$

For the remaining regions 16,17,18,20,21 and 22 global sections with the desired order of vanishing at R could not be found via computations up to $k = 12$.

The approach for proving rationality of the Seshadri constant applies for a certain class of polytopes. To describe this class we need to introduce the following terms.

Definition 5.6. Let $P \subseteq \mathbb{R}^n$ be a polytope and $v \in \mathbb{Z}^n$ a primitive vector. For a point $Q \in \partial P$ we define the length of P at Q with respect to v to be

$$\text{length}(P, Q, v) := \max \{|t| \mid t \in \mathbb{R} \text{ and } Q + tv \in P\}.$$

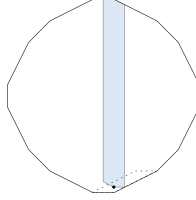
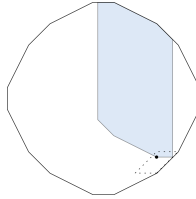
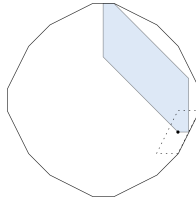
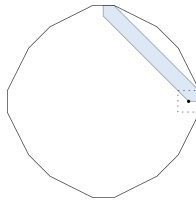
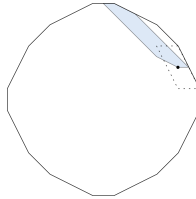
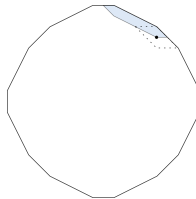
Definition 5.7. Let $P \subseteq \mathbb{R}^n$ be a polytope and $v \in \mathbb{Z}^n$ a primitive direction. Set

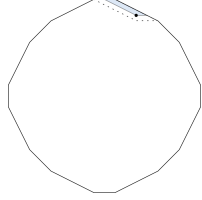
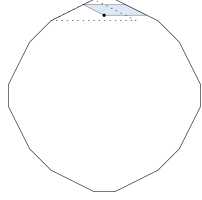
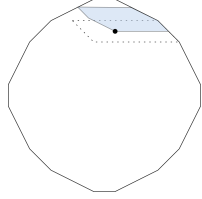
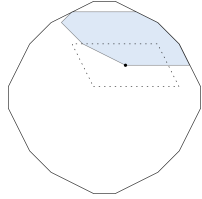
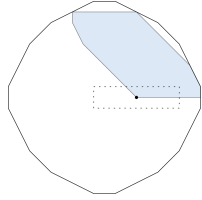
$$\begin{aligned} \text{vert}(P, v) &:= \{Q \in \partial P \mid Q = T + tv \text{ for some } T \in \text{vert}(P) \text{ and } t \in \mathbb{R}\} \\ &\cup \{Q \in \partial P \mid Q \in (P \cap (\text{sun}(P, v) + tv)) \text{ and } (P \cap (\text{sun}(P, v) + tv)) \\ &\quad \text{contains a vertex } T \in \text{vert}(P) \text{ for some } t \in \mathbb{R}_{\geq 0}\}. \end{aligned}$$

and call it the extended vertex set of P with respect to v .

Definition 5.8. Let $P \subseteq \mathbb{R}^2$ be a polygon. We call P zonotopally well-covered with respect to a primitive direction $v \in \mathbb{Z}^2$ if for all points $Q \in \text{vert}(P, v)$ the set

$$P(Q) := \begin{cases} P \cap (P + \text{length}(P, Q, v) \cdot v), & \text{if } Q \in \text{sun}(P, -v) \\ P \cap (P - \text{length}(P, Q, v) \cdot v), & \text{if } Q \in \text{sun}(P, v) \end{cases}$$

Region	Inequalities	Newton Polytope	Section	$\text{ord}_R(s)$
1	$0 \leq b \leq 1,$ $0 \leq a - 4b \leq 1$		$s(x, y) = x^a y^b$ $(x - 1)^{1-a+4b}$ $(y - 1)^b$ $(x^2 y - 1)^{9-2b}$	$10 - a + 3b$
2	$1 \leq b \leq 2,$ $1 \leq a - 3b \leq 2$		$s(x, y) = x^a y^b$ $(x - 1)^{2-a+3b}$ $(y - 1)^{2-b}$ $(x^2 y - 1)^{10-3b}$ $(xy - 1)^{3b-3}$	$11 - a + 2b$
3	$2 \leq b \leq 4,$ $4 \leq 2a - 5b \leq 6$		$s(x, y) = x^a y^b$ $(x - 1)^{3-a+\frac{5}{2}b}$ $(x^2 y - 1)^{7-\frac{3}{2}b}$ $(xy - 1)^{2+\frac{1}{2}b}$	$12 - a + \frac{3}{2}b$
4	$4 \leq b \leq 5,$ $4 \leq a - 2b \leq 5$		$s(x, y) = x^a y^b$ $(x - 1)^{5-a+2b}$ $(x^2 y - 1)^{5-b}$ $(xy - 1)^4$	$14 - a + b$
5	$1 \leq b \leq 2,$ $1 \leq a - 3b \leq 2$		$s(x, y) = x^a y^b$ $(x - 1)^{\frac{15}{2}-a+\frac{3}{2}b}$ $(y - 1)^{-\frac{5}{2}+\frac{1}{2}b}$ $(xy - 1)^{\frac{23}{2}-\frac{3}{2}b}$	$\frac{33}{2} - a + \frac{1}{2}b$
6	$7 \leq b \leq 8,$ $10 \leq a - b \leq 11$		$s(x, y) = x^a y^b$ $(x - 1)^{11-a+b}$ $(y - 1)$ $(xy - 1)^{8-b}$	$20 - a$

Region	Inequalities	Newton Polytope	Section	Value
7	$8 \leq b \leq 9,$ $18 \leq a \leq 19$		$s(x, y) = x^a y^b$ $(x - 1)^{19-a}$ $(y - 1)^{9-b}$	$28 - a - b$
8	$8 \leq b \leq 9,$ $a \leq 18,$ $-a + 4b \leq 18$		$s(x, y) = x^a y^b$ $(x - 1)^{19-a}$ $(y - 1)^{\frac{9}{2} + \frac{1}{4}a - b}$	$\frac{47}{2} - \frac{3}{4}a - b$
9	$7 \leq b \leq 8,$ $6 \leq a - b \leq 10$		$s(x, y) = x^a y^b$ $(x - 1)^{11-a+b}$ $(y - 1)^{-3 + \frac{1}{4}a - \frac{1}{4}b}$ $(xy - 1)^{8-b}$	$\frac{35}{2} - \frac{3}{4}a - \frac{1}{4}b$
10	$5 \leq b \leq 7,$ $5 \leq 2a - 3b \leq 13$		$s(x, y) = x^a y^b$ $(x - 1)^{\frac{15}{2} - a + \frac{3}{2}b}$ $(y - 1)^{-\frac{5}{8} + \frac{1}{4}a - \frac{3}{8}b}$ $(xy - 1)$ $(x^3 y^2 - 3xy + y + 1)^{\frac{7}{2} - \frac{1}{2}b}$	$\frac{119}{8} - \frac{3}{4}a + \frac{1}{8}b$
11	$4 \leq b \leq 5,$ $0 \leq a - 2b \leq 4$		$s(x, y) = x^a y^b$ $(x - 1)^{5-a+2b}$ $(x^2 y - 1)^{5-b}$ $(xy - 1)^{1 + \frac{3}{4}a - \frac{3}{2}b}$ $(x^3 y^2 - 3xy + y + 1)^{1 - \frac{1}{4}a + \frac{1}{2}b}$	$13 - \frac{3}{4}a + \frac{1}{2}b$

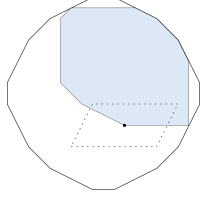
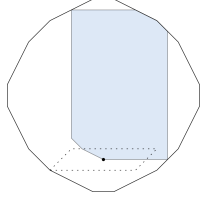
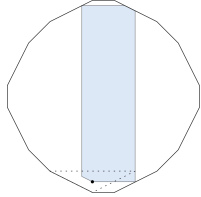
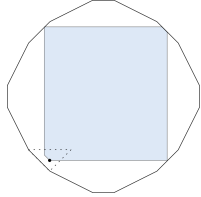
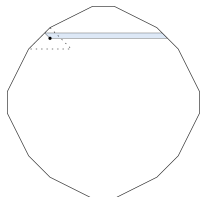
Region	Inequalities	Newton Polytope	Section	$\text{ord}_R(s)$
12	$2 \leq b \leq 4,$ $-4 \leq -2a + 5b$ $-2a + 5b \leq 4$		$s(x, y) = x^a y^b$ $(x - 1)^{3-a+\frac{5}{2}b}$ $(y - 1)^{\frac{1}{2}+\frac{1}{4}a-\frac{5}{8}b}$ $(x^2 y - 1)^{9-2b}$ $(xy - 1)$ $(x^3 y^2 - 3xy + y + 1)^{-1+\frac{1}{2}b}$	$\frac{23}{2} - \frac{3}{4}a + \frac{7}{8}b$
13	$1 \leq b \leq 2,$ $-3 \leq a - 3b \leq 1$		$s(x, y) = x^a y^b$ $(x - 1)^{2-a+3b}$ $(y - 1)^{\frac{3}{4}+\frac{1}{4}a-\frac{3}{4}b}$ $(x^2 y - 1)^{9-2b}$ $(xy - 1)^{-1+b}$	$\frac{43}{4} - \frac{3}{4}a + \frac{5}{4}b$
14	$5 \leq b \leq 7,$ $5 \leq 2a - 3b \leq 13$		$s(x, y) = x^a y^b$ $(x - 1)^{1-a+4b}$ $(y - 1)^{\frac{1}{4}a}$ $(x^2 y - 1)^{9-2b}$ $(xy - 1)^{-1+b}$	$10 - \frac{3}{4}a + 2b$
15	$1 \leq b \leq 2,$ $-a + b \leq 1,$ $-a + 3b \geq 3$		$s(x, y) = x^a y^b$ $(x - 1)^{2-a+3b}$ $(x^2 y - 1)^{9-2b}$ $(xy - 1)^{\frac{1}{2}+\frac{1}{2}a-\frac{1}{2}b}$	$\frac{23}{2} - \frac{1}{2}a + \frac{1}{2}b$
19	$7 \leq b \leq 8,$ $a - b \leq 6,$ $-a + 3b \leq 10$		$s(x, y) = x^a y^b$ $(x - 1)^{11-a+b}$ $(xy - 1)^{5+\frac{1}{2}a-\frac{3}{2}b}$	$16 - \frac{1}{2}a - \frac{1}{2}b$

TABLE 2. Sections s such that s^k are global sections of $\Gamma(X, \mathcal{O}_X(kD))$ that realize lower bounds for the order of vanishing ord_R for respective $k \in \mathbb{N}$

contains a zonotope $L_1 + \dots + L_\ell$, such that

$$\sum_{i=1}^{\ell} \text{length}_M(L_i) = \text{width}_u(P(Q)),$$

where $u \in N$ is primitive with $u(v) = 0$. The polygon P is zonotopally well-covered if it is with respect to some v .

Theorem 5.9. *Let X be a smooth projective toric surface and D an ample torus-invariant divisor on X with associated Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ for an admissible torus-invariant flag Y_\bullet . If the polytope $\Delta_{Y_\bullet}(D)$ is zonotopally well-covered, then the Seshadri constant at a general point $\varepsilon(X, D; R)$ is rational.*

Proof. Since all input data is torus-invariant, the Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ is isomorphic to the polytope $P = P_D$ for any admissible torus-invariant flag Y_\bullet . By assumption, this polytope is zonotopally well-covered, so let $v = (v_1, v_2) \in \mathbb{Z}^2$ be its associated primitive direction. Consider the flag $Y'_\bullet: X \supseteq C \supseteq \{R\}$, where C is the curve given by the local equation $x^{v_1}y^{v_2} - 1 = 0$ and $R = (1, 1)$ is a general point on C . Then the shifting process explained in Section 3 yields the Newton–Okounkov body $\Delta_{Y'_\bullet}(D)$ with respect to this new flag. By Corollary 3.8 this process relates the Newton–Okounkov bodies via a piecewise linear homeomorphism $\Psi: \Delta_{Y_\bullet}(D) \xrightarrow{\sim} \Delta_{Y'_\bullet}(D)$.

We show that the Newton–Okounkov function $\varphi'_R: \Delta_{Y'_\bullet}(D) \rightarrow \mathbb{R}$ that comes from the geometric valuation ord_R satisfies $\varphi'_R(a', b') = a' + b'$ for all vertices $T = (a', b') \in \text{vert}(\Delta_{Y'_\bullet}(D))$. Together with the facts that φ'_R is concave and has $a' + b'$ as an upper bound it follows that $\varphi'_R(a', b') = a' + b'$ on the entire Newton–Okounkov body. Rationality of the integral $\int_{\Delta_{Y'_\bullet}(D)} \varphi'_R$ yields rationality of the Seshadri constant $\varepsilon(X, D; R)$.

Let $Q_{Y_\bullet} = \Psi^{-1}(0, 0)$ be the vertex of $\Delta_{Y_\bullet}(D)$ that gets mapped to the origin. Choose the torus-invariant flag Y_\bullet for $\Delta_{Y_\bullet}(D)$ such that Q_{Y_\bullet} lies at the opposite side of $\Delta_{Y_\bullet}(D)$ with respect to Y_\bullet .

Let $T = (a', b') \in \text{vert}(\Delta_{Y'_\bullet}(D))$ be a vertex and let $Q = (a, b) = \Psi^{-1}(T)$ denote its preimage in $\Delta_{Y_\bullet}(D)$. Then $Q \in \text{vert}(\Delta_{Y_\bullet}(D), v)$. Since $\Delta_{Y_\bullet}(D)$ is zonotopally well-covered, there exist line segments L_1, \dots, L_ℓ with

$$\sum_{i=1}^{\ell} \text{length}_M(L_i) = \text{width}_u(P(Q)) = \text{MV}(P(Q), \text{NP}(x^{v_1}y^{v_2} - 1)),$$

such that $(L_1 + \dots + L_\ell) \subseteq P(Q)$.

Consider the section that is locally given as

$$s(x, y) = \begin{cases} x^a y^b \cdot g_1 \cdot \dots \cdot g_\ell & \text{if } Q = \Psi^{-1}(0, b') \text{ for some } b', \\ x^a y^b \cdot (x^{v_1} y^{v_2} - 1)^{\text{length}(\Delta_{Y_\bullet}(D), Q, v)} & \text{if } Q = \Psi^{-1}(a', 0) \text{ for some } a', \\ x^a y^b \cdot (x^{v_1} y^{v_2} - 1)^{\text{length}(\Delta_{Y_\bullet}(D), Q, v)} \cdot g_1 \cdot \dots \cdot g_\ell & \text{otherwise,} \end{cases}$$

where g_i denotes a section associated to L_i .

By construction, the following holds for the section s .

(1) The Newton polytope $\text{NP}(s)$ fits inside $\Delta_{Y_\bullet}(D)$ and therefore s^k is a global section of $\Gamma(X, \mathcal{O}_X(kD))$ for some suitable $k \in \mathbb{N}$.

(2) The section s^k is mapped to $Q = \frac{1}{k} \text{val}_{Y_\bullet}(s^k)$ when computing the Newton-Okounkov body $\Delta_{Y_\bullet}(D)$.

(3) When computing the Newton-Okounkov body $\Delta_{Y'_\bullet}(D)$ it is mapped to

$$\begin{aligned} & \frac{1}{k} \text{val}_{Y_\bullet}(s^k) \\ &= (\text{ord}_C(s), \text{ord}_R(s_1)) \\ &= \begin{cases} \left(0, \sum_{i=1}^{\ell} \text{length}_M(L_i)\right) & \text{if } Q = \Psi^{-1}(0, b') \text{ for some } b', \\ (\text{length}(\Delta_{Y_\bullet}(D), Q, v), 0) & \text{if } Q = \Psi^{-1}(a', 0) \text{ for some } a', \\ \left(\text{length}(\Delta_{Y_\bullet}(D), Q, v), \sum_{i=1}^{\ell} \text{length}_M(L_i)\right) & \text{otherwise,} \end{cases} \\ &= \begin{cases} (0, \text{width}_u(P(Q))) & \text{if } Q = \Psi^{-1}(0, b') \text{ for some } b', \\ (\text{length}(\Delta_{Y_\bullet}(D), Q, v), 0) & \text{if } Q = \Psi^{-1}(a', 0) \text{ for some } a', \\ (\text{length}(\Delta_{Y_\bullet}(D), Q, v), \text{width}_u(P(Q))) & \text{otherwise,} \end{cases} \\ &= T. \end{aligned}$$

(4) For the order of vanishing at R we have

$$\begin{aligned} & \frac{1}{k} \text{ord}_R(s^k) \\ &= \text{ord}_R(s) \\ &= \begin{cases} \sum_{i=1}^{\ell} \text{length}_M(L_i) & \text{if } Q = \Psi^{-1}(0, b') \text{ for some } b' \\ \text{length}(\Delta_{Y_\bullet}(D), Q, v) & \text{if } Q = \Psi^{-1}(a', 0) \text{ for some } a', \\ \text{length}(\Delta_{Y_\bullet}(D), Q, v) + \sum_{i=1}^{\ell} \text{length}_M(L_i) & \text{otherwise,} \end{cases} \\ &= a' + b'. \end{aligned}$$

Alltogether this yields $\varphi'_R(a', b') = a' + b'$ for all vertices $T = (a', b') \in \text{vert}(\Delta_{Y'_\bullet}(D))$ and therefore the rationality of the Seshadri constant $\varepsilon(X, D; R)$. \square

Example 5.10. To illustrate the proof we stick to the previous Example 5.5. The polytope $\Delta_{Y_\bullet}(D)$ is zonotopally well-covered with respect to $v = (-1, 0)$. Consider, for instance the vertex $T = (7, 5) \in \Delta_{Y'_\bullet}(D)$. Its preimage under the piecewise linear isomorphism Ψ is $Q = (2, 1) \in \Delta_{Y_\bullet}(D)$ and $\text{length}(\Delta_{Y_\bullet}(D), Q, v) = 7$. A global section which is mapped to Q and T respectively, is

$$\begin{aligned} s(x, y) &= x^a y^b \cdot (x^{v_1} y^{v_2} - 1)^{\text{length}(\Delta_{Y_\bullet}(D), Q, v)} \cdot g_1 \\ &= xy^2 \cdot (x - 1)^7 \cdot (xy^2 - 1)^5 \end{aligned}$$

as seen in Figure 29 with $\text{ord}_R(s) = 7 + 5 = 12$.

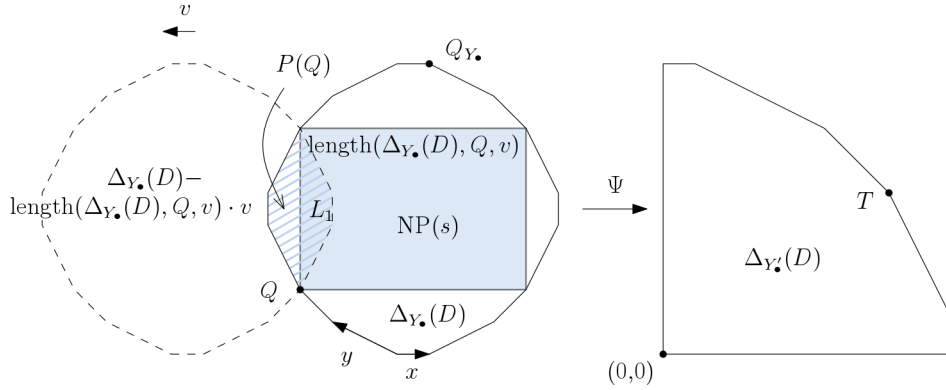


FIGURE 29. The setup of the proof of Theorem 5.9 in the context of Example 5.5

Remark 5.11. The property of being centrally-symmetric is not sufficient for being zonotopally well-covered. Consider for instance the polytope

$$P = \text{conv}((0, 0), (2, 1), (1, 3), (-1, 2)) \subseteq \mathbb{R}^2$$

in Figure 30 and the direction $v = (-1, 0)$. Then the point $Q = (-\frac{1}{2}, 1) \in \text{vert}(P, v)$ and P has length $\text{length}(P, Q, v) = \frac{5}{2}$ at Q with respect to v . The intersection $P(Q) = P \cap (P + \frac{5}{2} \cdot (-1, 0))$ is just a line segment L whose lattice length is $\text{length}_{\mathbb{Z}^2}(L) = \frac{1}{2}$. But on the other hand we have $\text{width}_u(P(Q)) = 1 > \frac{1}{2}$ for $u = (0, 1)$.

It remains to argue why any other direction $v \in \mathbb{Z}^2$ will also fail. If we interpret P as the Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ for some completely toric situation X, D, Y_\bullet then the shifting process by the vector $v = (-1, 0)$ yields the polytope on right in Figure 30 as the Newton–Okounkov body $\Delta_{Y'_\bullet}(D)$ for the adjusted flag Y'_\bullet , where Y'_1 is the curve determined by v and $Y'_2 = R = (1, 1)$. Consider the Newton–Okounkov function $\varphi'_R: \Delta_{Y'_\bullet}(D) \rightarrow \mathbb{R}$. Since $\varphi'_R(a', b') \leq a' + b'$ and $\max_{(a', b') \in \Delta_{Y'_\bullet}(D)} \varphi'_R(a', b')$

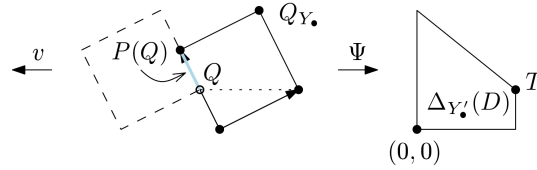


FIGURE 30. An instance of a centrally-symmetric polytope P that is not zonotopally well-covered

is independent of the flag, this yields that $\max \varphi'_R \leq \frac{7}{2}$. A straight forward computation shows that any primitive direction $v \in \mathbb{Z}^2$ with $\|v\| > 1$ results in the vertex $(0, b') \in \Delta_{Y'}(D)$ with $b' > \frac{7}{2}$ which is a contradiction to the above.

Although P is not the polytope of an ample divisor on a smooth surface, it can be used as a starting point to construct such an example: The minimal resolution $\pi: X_P^* \rightarrow X_P$ has a centrally-symmetric fan. There is a “centrally-symmetric” ample \mathbb{Q} -divisor on X_P^* near the nef divisor π^*D . Now scale up the resulting rational polygon to a lattice polygon.

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