Introduction to Grothendieck's Section Conjecture in Anabelian Geometry

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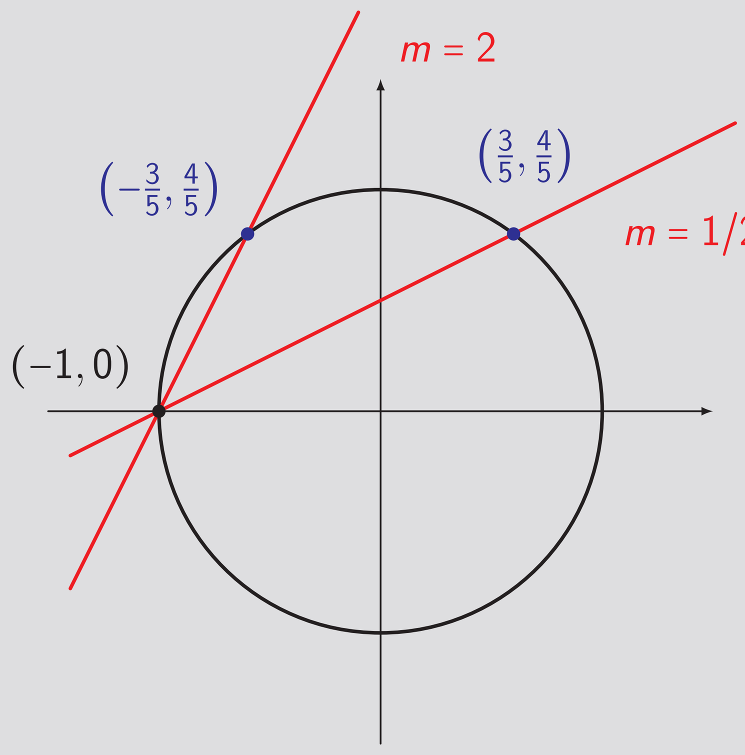
Problem: Understanding k-rational points of curves

Classically, a diophantine problem is of the following kind: We are given a polynomial equation

$$f(\underline{x}) = f(x_1, \dots, x_n) = 0$$

where we can assume that $f(\underline{x}) \in \mathbb{Z}[\underline{x}]$. We look for solutions $\underline{x} \in R$ where R is something like $\mathbb{Z}, \mathbb{Z}[1/n], \mathbb{Q}$, a number field or a p-adic local field.

An easy, yet interesting, example is finding the rational solutions of $x^2 + y^2 = 1$. This means finding the rational points on the unit circle. This can be done via parametrization as illustrated on the right, giving us infinitely many rational points. For sufficiently complicated examples, this method will not work.
In fact, we have the following result:



Mordell Conjecture (Faltings, '83). Let X be a smooth proper curve of genus at least 2 over a number field k. Then $\#X(k) < \infty$.

This poster introduces an approach to rational points using fundamental groups.

We want to introduce the notion of fundamental group of a scheme but the Zariski topology is too coarse for the notion of path to be well-behaved. However, there is a good notion of covering space given by finite étale morphisms (i.e. morphisms f which are of finite presentation, flat and such that $\Omega_f = 0$).

The Algebraic Fundamental Group

Let X be a connected scheme. We denote by

$$\mathrm{Fin\acute{E}t}(X) = \{\pi : Y \to X \mid \pi \text{ finite étale morphism}\}$$

the category of finite étale coverings of X. Let $\overline{x}\colon \mathrm{Spec}(\Omega) \to X$ be a geometric point of X. The *fiber functor at* $\overline{x}$ is given by

$$\mathrm{Fib}_{\overline{x}}: \quad \mathrm{Fin\acute{E}t}(X) \to \mathrm{FinSets}, \quad (Y \xrightarrow{\pi} X) \mapsto \pi^{-1}(\overline{x}) = Y \times_X \mathrm{Spec}\,\Omega$$

Definition. We define the *étale fundamental group of X with base point $\overline{x}$* to be $\pi_1^{\acute{e}t}(X, \overline{x}) := \mathrm{Aut}(\mathrm{Fib}_{\overline{x}})$. This is a profinite group and $\pi_1^{\acute{e}t}$ is functorial.

The fiber functor induces the following equivalence of categories

$$\mathrm{Fin\acute{E}t}(X) \longleftrightarrow \pi_1^{\acute{e}t}(X, \overline{x})\text{-FinSets}.$$

Properties:

- $\mathrm{Fib}_{\overline{x}}$ is pro-representable by $\widetilde{X} = (X_\alpha)_{\alpha \in A}$, where $(X_\alpha)_{\alpha \in A}$ is a filtered inverse system of Galois covers of X. It follows that

$$\pi_1^{\acute{e}t}(X, \overline{x}) = \varprojlim_{\alpha \in A} \mathrm{Aut}(X_\alpha)^{\mathrm{opp}}.$$

- Given two geometric points $\overline{x}, \overline{y}$ of X, there is an isomorphism of fiber functors $\gamma\colon \mathrm{Fib}_{\overline{x}} \xrightarrow{\cong} \mathrm{Fib}_{\overline{y}}$ inducing

$$\pi_1^{\acute{e}t}(X, \overline{y}) \xrightarrow{\cong} \pi_1^{\acute{e}t}(X, \overline{x}), \quad \phi \mapsto \gamma^{-1} \circ \phi \circ \gamma.$$

We define an *étale path on X between $\overline{x}$ and $\overline{y}$* to be an element of $\pi_1^{\acute{e}t}(X; \overline{x}, \overline{y}) := \mathrm{Isom}(\mathrm{Fib}_{\overline{x}}, \mathrm{Fib}_{\overline{y}})$.

Example: This generalizes the following two theories.

- *Galois Theory:* Let k be a field and k^{sep} a separable closure of k (defining a geometric point of $\mathrm{Spec}(k)$). Let $\mathrm{Gal}_k := \mathrm{Gal}(k^{\mathrm{sep}}/k)$. Then

$$\pi_1^{\acute{e}t}(\mathrm{Spec}(k), \overline{x}) = \mathrm{Gal}_k\,.$$

- *Topological fundamental group:* Let $X/\mathbb{C}$ be a variety, $x \in X(\mathbb{C})$. We have the following equivalences of categories:

$$\mathrm{Fin\acute{E}t}(X) \overset{\text{GAGA}}{\longleftrightarrow} \mathrm{FinCov}(X(\mathbb{C})) \overset{\text{induced by fiber functor}}{\longleftrightarrow} \widehat{\pi_1^{\mathrm{top}}(X(\mathbb{C}), x)}\text{-FinSets},$$

where for a group G we denote by $\widehat{G}$ the profinite completion of G. Hence,

$$\pi_1^{\acute{e}t}(X, x) \cong \widehat{\pi_1^{\mathrm{top}}(X(\mathbb{C}), x)}.$$

Now if K/k is an extension of algebraically closed fields and X/k is a proper integral scheme, $X_K \to X$ induces an isomorphism of fundamental groups. E.g. if k is a number field, we have $\pi_1^{\acute{e}t}(X_{\overline{k}}, \overline{x}) \cong \widehat{\pi_1^{\mathrm{top}}(X(\mathbb{C}), \overline{x})}$. We call this the *geometric fundamental group*. Clearly, it will not suffice to study this for an arithmetic description of X as it only contains topological information about $X(\mathbb{C})$, e.g. the genus in the case of curves. However, it turns out that $\pi_1^{\acute{e}t}$ of X/k is essentially described by the geometric fundamental group and the absolute Galois group of k:

Where Geometry and Arithmetic meet: Homotopy exact sequence

Let k be a field and X/k a geometrically connected variety. Fix a separable algebraic closure k^{sep}. Let $\overline{X} = X \times_k k^{\mathrm{sep}}$ and choose a geometric point $\overline{x} \in \overline{X}$. By functoriality, $X \to \mathrm{Spec}(k)$ induces the following short exact sequence:

$$\pi_1(X/k): \quad 1 \longrightarrow \underbrace{\pi_1^{\acute{e}t}(\overline{X}, \overline{x})}_{\text{GEOMETRY}} \longrightarrow \pi_1^{\acute{e}t}(X, \overline{x}) \longrightarrow \underbrace{\mathrm{Gal}(k^{\mathrm{sep}}/k)}_{\text{ARITHMETIC}} \longrightarrow 1.$$

Let $a : \mathrm{Spec}(k) \to X$ be a rational point. By functoriality, this induces a map from Gal_k to $\pi_1^{\acute{e}t}(X, \overline{a})$. Choosing an étale path $\gamma \in \pi_1^{\acute{e}t}(\overline{X}; \overline{x}, \overline{a})$, which is unique up to $\pi_1^{\acute{e}t}(\overline{X}, \overline{x})$-conjugation, induces a section of $\pi_1(X/k)$, namely

$$s_a: \ \mathrm{Gal}_k \xrightarrow{\sigma_a} \pi_1^{\acute{e}t}(X, \overline{a}) \xrightarrow{\gamma(\ )\gamma^{-1}} \pi_1^{\acute{e}t}(X, \overline{x}).$$

Define

$$\mathscr{S}_{\pi_1(X/k)} := \{\text{sections of } \pi_1(X/k)\} / \pi_1^{\acute{e}t}(\overline{X}, \overline{x})\text{-conjugation}.$$

Given a section (e.g. if $\overline{x} \in X(k)$), this is in bijection to the (non-abelian) cohomology set

$$H^1(\mathrm{Gal}_k, \pi_1^{\acute{e}t}(\overline{X}, \overline{x})),$$

which classifies Gal_k-equivariant $\pi_1^{\acute{e}t}(\overline{X}, \overline{x})$-torsors.

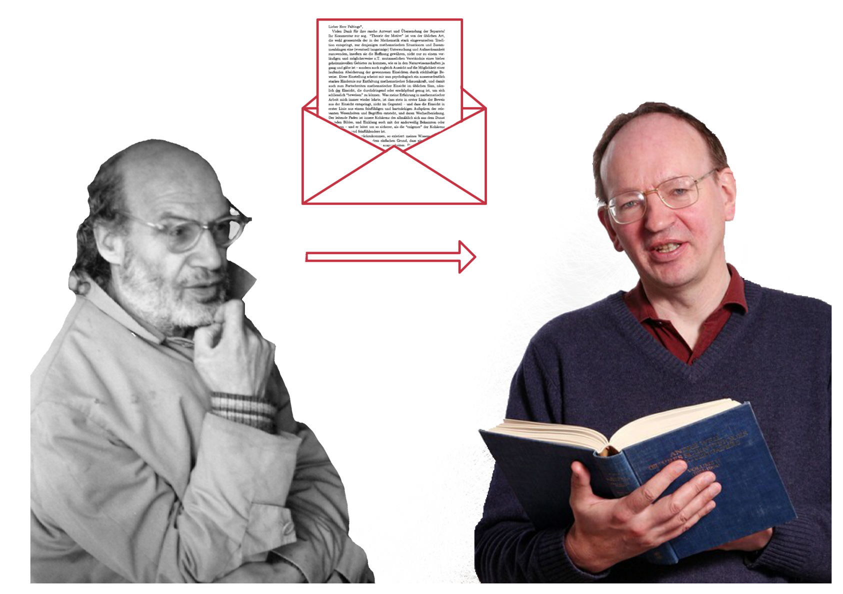
The Section Conjecture

We have the *profinite Kummer map*

$$\begin{array}{rccccc} \kappa: & X(k) & \longrightarrow & \mathscr{S}_{\pi_1(X/k)} & = & H^1(\mathrm{Gal}_k, \pi_1^{\acute{e}t}(\overline{X}, \overline{x})) \\ & a & \mapsto & [s_a] & = & [\pi_1^{\acute{e}t}(\overline{X}; \overline{x}, \overline{a})]. \end{array}$$

Conjecture (Grothendieck's Section Conjecture). Let k be a finitely generated field extension of $\mathbb{Q}$ and X a smooth, projective, geometrically connected curve of genus at least 2 over k. Then κ is a bijection.

This conjecture was first stated by Grothendieck in his 1983 "anabelian" letter to Faltings (see [1]) in which he also explained how to deduce injectivity of the profinite Kummer map from the Mordell–Weil theorem. The surjectivity is still open. For an overview of the state of the art on this conjecture, see [3].



The Section Conjecture is part of the "anabelian program" since the condition on the genus of the curve X is equivalent to $\pi_1^{\acute{e}t}(X)$ being non-abelian. It is conjectured that if the fundamental group of a scheme X is "sufficiently anabelian", X can be reconstructed up to isomorphism from $\pi_1^{\acute{e}t}(X)$. An arithmetic example of this is the Theorem of Neukirch–Uchida which proves this for number fields.

References

[1] A. Grothendieck, Brief an Faltings (27.06.1983), in: *Geometric Galois Action 1* (ed. L. Schneps, P. Lochak), LMS Lecture Notes 242, Cambridge 1997, 49–58.

[2] A. Grothendieck, *Revêtements étales et groupe fondamental (SGA 1)*, Lecture Notes in Mathematics, Vol. 224, Springer, 1971.

[3] J. Stix, *Rational points and arithmetic of fundamental groups*, Volume 2054 of Lecture Notes in Mathematics, Springer, Heidelberg, 2013.

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