Regularization of an inverse nonlinear parabolic problem with time-dependent coefficient and locally Lipschitz source term

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Abstract

We consider a backward problem of finding a function u satisfying a nonlinear parabolic equation in the form $u_t + a(t)Au(t) = f(t, u(t))$ subject to the final condition $u(T) = \varphi$. Here A is a positive self-adjoint unbounded operator in a Hilbert space H and f satisfies a locally Lipschitz condition. This problem is ill-posed. Using quasi-reversibility method, we shall construct a regularized solution u_ε from the measured data a_ε and φ_ε. We show that the regularized problem are well-posed and that their solutions converge to the exact solutions. Error estimate is given.

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1. Introduction

Let $(H, \|\cdot\|)$ be a Hilbert space with the inner product $(\cdot, \cdot)$. Let A be a positive self-adjoint operator defined on a dense subspace $D(A) \subset H$ such that $-A$ generates a compact contraction semi-group $S(t)$ on H. Let $f : [0, T] \times H \to H$ satisfy the locally Lipschitz condition: for each $M > 0$, there exists $k(M) > 0$ such that

$$\|f(t, u) - f(t, v)\| \leq k(M) \|u - v\| \text{ if } \max\{\|u\|, \|v\|\} \leq M. \tag{1}$$

We shall consider a backward problem of finding a function $u : [0, T] \to H$ such that

$$\begin{aligned} u_t + a(t)Au(t) &= f(t, u(t)), \quad 0 < t < T, \\ u(T) &= \varphi, \end{aligned} \tag{2}$$

where $a \in C([0, T])$ is a given real-valued function and $\varphi \in H$ is a prescribed final value.

This nonlinear nonhomogeneous problem is severely ill-posed. In fact, the problem is extremely sensitive to measurement errors (see, e.g., [2]). The final data is usually the result of discrete experimental measurements and is subject to error. Hence, a solution corresponding to the data does not

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always exist, and in the case of existence, does not depend continuously on the given data. This, of course, shows that a naturally numerical treatment is impossible. Thus one has to resort to a regularization.

The backward problem (2) has a long history. The linear homogeneous case $f = 0$ has been considered by many authors such as quasi-reversibility method [7, 8, 6, 10, 1], quasi-boundary value method [4, 5]. The problem with constant coefficient and nonlinear source term, i.e.

$$\begin{aligned} u_t + Au(t) &= f(t, u(t)), \ \ 0 < t < T, \\ u(T) &= \varphi, \end{aligned} \tag{3}$$

was studied in [3, 12, 13, 14]. However, in these papers, the source function f is assumed to be globally Lipschitz, that is

$$\|f(t, u) - f(t, v)\| \le k\|u - v\|$$

where k is independent of t, u. Recently, in [15], a regularization method for locally Lipschitz source term has been established under an extra condition on the source term:

$$\text{There exists a constant } L \ge 0, \text{ such that } \langle f(t, u) - f(t, v), u - v\rangle + \|u - v\|^2 \ge 0.$$

This condition holds for the source $f(u) = u\,\|u\|_H^2$ (see [15]). However, it is not satisfied in several cases, for example, $f(u) = au - bu^3$ ($b > 0$) of the Ginzburg-Landau equation. Hence, another regularization method which can be applied to any locally Lipschitz source term is of interests. In this paper, we shall assume that the source term f is locally Lipschitz with respect to u (i.e. f satisfies (1)). Our main idea is approximating the function f by a sequence f_ε of Lipschitz functions

$$\|f_\varepsilon(t, u) - f_\varepsilon(t, v)\| \le k_\varepsilon\|u - v\|.$$

Then, we use the results in [12, 14] to approximate problem (3) by the following problem

$$\begin{aligned} \frac{d}{dt}u^\varepsilon(t) + A_\varepsilon u^\varepsilon(t) &= B(\varepsilon, t) f_\varepsilon(t, u^\varepsilon(t)), \quad t \in [0, T], \\ u^\varepsilon(T) &= \varphi \end{aligned} \tag{4}$$

where A_ε, $B(\varepsilon, t)$ are defined appropriately.

When the perturbed coefficient a is time-dependent, the problems turns to be more complicated. Indeed, the strategies used for constant coefficient cannot be applied to the time-dependent coefficient case. The problem with time-dependent coefficient has been recently investigated in [9]. However, the methods proposed in [9] can be merely applied either for zero source with perturbed time-dependent coefficient or for globally Lipschitz source with unperturbed time-dependent coefficient. We would like to emphasize that our regularization method for constant coefficient also works for unperturbed time-dependent coefficient.

The paper is organized as follows. In Section 2, we shall investigate a regularization method for the case of constant coefficient $a \equiv 1$. In particular, we shall give precise formulas of A_ε, $B(\varepsilon, t)$ and $f_\varepsilon(t, v)$; show that the regularized problem (4) is well-posed and prove the convergence of u^ε to the exact solution in $C([0, T]; H)$ with explicit error estimates. Section 3 provides a regularization method for perturbed time-dependent coefficient $a(t)$.

2. Regularization of backward parabolic problem with constant coefficient

2.1. The well-posedness of the regularized problem (4)

We shall first give the precise formula of the operator $S(t)$. Assume that A is a positive self-adjoint operator in the separable Hilbert space $(H, (\cdot,\cdot))$ and 0 is in its resolvent set. Since A^{-1} is a compact self-adjoint operator, there is an orthonormal eigenbasis $\{\phi_n\}_{n=1}^{\infty}$ of H corresponding to a sequence of its eigenvalues $\{\lambda_n^{-1}\}_{n=1}^{\infty}$ in which

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... \lim_{n\to\infty} \lambda_n = \infty.$$

Thus $A^{-1}\phi_n = \lambda_n^{-1}\phi_n$ and $A\phi_n = \lambda_n\phi_n$ for each $n \geq 1$. The compact contraction semi-group $S(t)$ corresponding to A is

$$S(t)v = \sum_{n=1}^{\infty} e^{-t\lambda_n}(\phi_n, v)\phi_n, \ v \in H.$$

Problem (3) can be written in the language of semi-group as follows.

$$u(t) = S(t-T)\varphi - \int_t^T S(t-s)f(s, u(s))\,\mathrm{d}s. \tag{5}$$

For each $\varepsilon > 0$, we define the bounded operator

$$A_\varepsilon(v) \quad = \quad -\frac{1}{T}\sum_{n=1}^{\infty} \ln(\varepsilon + e^{-T\lambda_n})(\phi_n, v)\phi_n. \tag{6}$$

The compact contraction semi-group $S_\varepsilon(t)$ corresponding to A_ε is

$$S_\varepsilon(t)v = \sum_{n=1}^{\infty} \left(\varepsilon + e^{-T\lambda_n}\right)^{\frac{t}{T}} (\phi_n, v)\phi_n, \ v \in H.$$

Obviously, (4) can be written as

$$u^\varepsilon(t) = S_\varepsilon(t-T)\varphi - \int_t^T S_\varepsilon(t-s)B(\varepsilon, s)f_\varepsilon(s, u^\varepsilon(s))\,\mathrm{d}s, \tag{7}$$

For each $t \leq T$, define by $B(\varepsilon, t)$ the bounded operator

$$B(\varepsilon, t) := S_\varepsilon(t-T)S(T-t).$$

The operator $B(\varepsilon, t)$ can be written explicitly as

$$B(\varepsilon, t)(v) = \sum_{n=1}^{\infty}(1 + \varepsilon e^{T\lambda_n})^{\frac{t}{T}-1}(\phi_n, v)\phi_n, \ v \in H. \tag{8}$$

In particular,

$$\begin{aligned} B(\varepsilon, t)\phi_n &= S_\varepsilon(t-T)S(T-t)\phi_n = S_\varepsilon(t-T)\left(e^{-(T-t)\lambda_n}\phi_n\right) \\ &= \left(\varepsilon + e^{-T\lambda_n}\right)^{\frac{t-T}{T}} e^{-(T-t)\lambda_n}\phi_n = (\varepsilon e^{T\lambda_n} + 1)^{\frac{t-T}{T}}\phi_n, \ \forall n \geq 1. \end{aligned}$$

Our later calculations will be represented via operators $S_\varepsilon(t)$ and $B(\varepsilon, t)$. We shall need some upper bounds of these operators.

Lemma 1. *Let* $0 \le t \le T$. *Then* $S_\varepsilon(-t)$ *and* $B(\varepsilon, t)$ *are bounded operators and*

$$\|S_\varepsilon(-t)\| \le \varepsilon^{-\frac{t}{T}}, \qquad \|B(\varepsilon, t)\| \le 1.$$

Moreover,

$$\|[B(\varepsilon, t) - I]\,\phi_n\| \le \varepsilon e^{T\lambda_n}, \forall n \ge 1.$$

Proof. For each $n \ge 1$, one has

$$\begin{aligned} \|S_\varepsilon(t)\phi_n\| &= \left(\varepsilon + e^{-T\lambda_n}\right)^{-\frac{t}{T}} \le \varepsilon^{-\frac{t}{T}}, \\ \|B(\varepsilon, t)\phi_n\| &= (1 + \varepsilon e^{T\lambda_n})^{\frac{t}{T}-1} \le 1 \\ \|[I - B(\varepsilon, t)]\,\phi_n\| &= 1 - (1 + \varepsilon e^{T\lambda_n})^{\frac{t}{T}-1} \\ &\le 1 - (1 + \varepsilon e^{T\lambda_n})^{-1} \le \varepsilon e^{T\lambda_n}. \end{aligned}$$

The desired result follows. □

Next, we define an approximation f_ε of f. Recall that $f : [0, T] \times H \to H$ satisfies the locally Lipschitz condition (1):

For each $M > 0$, there exists $k(M) > 0$ such that $\|f(t, u) - f(t, v)\| \le k(M)\,\|u - v\|$ if $\max\{\|u\|, \|v\|\} \le M$.

It is obvious that the function k is increasing on $[0, \infty)$. We can choose a set $\{M_\varepsilon > 0\}_{\varepsilon>0}$ satisfying $\lim\limits_{\varepsilon\to 0^+} M_\varepsilon = \infty$ and $k(M_\varepsilon) \le \ln(\ln(\varepsilon^{-1}))/(4T)$. Define

$$f_\varepsilon(t, v) = f\left(t, \min\left\{\frac{M_\varepsilon}{\|v\|}, 1\right\} v\right), \qquad \forall (t, v) \in [0, T] \times H, \tag{9}$$

in particular $f_\varepsilon(t, 0) = f(t, 0)$. With this definition, we claim that f_ε is a Lipschitz function. In fact, we have

Lemma 2. *For* $\varepsilon > 0$, $t \in [0, T]$ *and* $v_1, v_2 \in H$, *one has*

$$\|f_\varepsilon(t, v_1) - f_\varepsilon(t, v_2)\| \le k_\varepsilon\,\|v_1 - v_2\|,$$

where $k_\varepsilon = 2k(M_\varepsilon) \le \ln(\ln(\varepsilon^{-1}))/(2T)$.

Proof. Due to the continuity, it is enough to prove Lemma 2 for non-zero vectors v_1, v_2. We can assume that $\|v_1\| \ge \|v_2\| > 0$. Using the locally Lipschitz property of f, one has

$$\begin{aligned} \|f_\varepsilon(t, v_1) - f_\varepsilon(t, v_2)\| &= \left\| f\left(t, \min\left\{\frac{M_\varepsilon}{\|v_1\|}, 1\right\} v_1\right) - f\left(t, \min\left\{\frac{M_\varepsilon}{\|v_2\|}, 1\right\} v_2\right)\right\| \\ &\le k(M_\varepsilon)\left\|\min\left\{\frac{M_\varepsilon}{\|v_1\|}, 1\right\} v_1 - \min\left\{\frac{M_\varepsilon}{\|v_2\|}, 1\right\} v_2\right\|. \end{aligned}$$

It remains to show that

$$\left\|\min\left\{\frac{M_\varepsilon}{\|v_1\|}, 1\right\} v_1 - \min\left\{\frac{M_\varepsilon}{\|v_2\|}, 1\right\} v_2\right\| \le 2\,\|v_1 - v_2\|.$$

This inequality is trivial if $M_\varepsilon \geq \|v_1\| \geq \|v_2\|$. When $\|v_1\| \geq \|v_2\| \geq M_\varepsilon$, one has

$$\begin{aligned}
\left\|\frac{M_\varepsilon}{\|v_1\|}v_1 - \frac{M_\varepsilon}{\|v_2\|}v_2\right\| &= M_\varepsilon\left\|\frac{v_1 - v_2}{\|v_1\|} + \frac{\|v_2\| - \|v_1\|}{\|v_1\|\,.\,\|v_2\|}v_2\right\| \\
&\leq M_\varepsilon\left(\left\|\frac{v_1 - v_2}{\|v_1\|}\right\| + \left\|\frac{\|v_2\| - \|v_1\|}{\|v_1\|\,.\,\|v_2\|}v_2\right\|\right) \\
&= \frac{M_\varepsilon}{\|v_1\|}\left(\|v_1 - v_2\| + |\|v_2\| - \|v_1\||\right) \leq 2\,\|v_1 - v_2\|\,.
\end{aligned}$$

Finally, if $\|v_1\| \geq M_\varepsilon \geq \|v_2\|$ then

$$\begin{aligned}
\left\|\frac{M_\varepsilon}{\|v_1\|}v_1 - v_2\right\| &= \left\|\frac{M_\varepsilon - \|v_1\|}{\|v_1\|}v_1 + v_1 - v_2\right\| \\
&\leq \left\|\frac{M_\varepsilon - \|v_1\|}{\|v_1\|}v_1\right\| + \|v_1 - v_2\| \\
&= |M_\varepsilon - \|v_1\|| + \|v_1 - v_2\| \leq 2\,\|v_1 - v_2\|\,.
\end{aligned}$$

Here we have used the inequality $|M_\varepsilon - \|v_1\|| \leq |\|v_2\| - \|v_1\|| \leq \|v_1 - v_2\|$. □

We now study the existence, the uniqueness and the stability of a (weak) solution of problem (4).

Theorem 1. *Let $\varepsilon > 0$. For each $\varphi \in H$, problem (4) has a unique solution $u^\varepsilon \in C([0,T];H)$. Moreover, the solutions depend continuously on the data in the sense that if u_j^ε is the solution corresponding to φ_j, $j = 1, 2$, then*

$$\|u_1^\varepsilon(t) - u_2^\varepsilon(t)\| \leq \varepsilon^{\frac{t-T}{T}} e^{k_\varepsilon(T-t)}\|\varphi_1 - \varphi_2\|.$$

Proof. **Step 1: Uniqueness**

Fix $\varphi \in H$. For each $w \in C([0,T];H)$, define by

$$F(w)(t) := S_\varepsilon(t-T)\varphi - \int_t^T S_\varepsilon(t-s)B(\varepsilon,s)f_\varepsilon(s,w(s))\,\mathrm{d}s.$$

It is sufficient to show that F has a unique fixed point in $C([0,T];H)$. This fact will be proved by contraction principle.

We claim by induction with respect to $m = 1, 2, ...$ that, for all $w, v \in C([0,T];H)$,

$$\|F^m(w)(t) - F^m(v)(t)\| \leq \left(\frac{k_\varepsilon}{\varepsilon}\right)^m \frac{(T-t)^m}{m!}|||w(s) - v(s)|||, \tag{10}$$

where $|||.|||$ is the sup norm in $C([0,T];H)$. For $m = 1$, using lemmas 1 and 2, we have

$$\begin{aligned}
\|F(w)(t) - F(v)(t)\| &= \left\|\int_t^T S_\varepsilon(t-s)B(\varepsilon,s)\left[f_\varepsilon(s,w(s)) - f_\varepsilon(s,v(s))\right]\mathrm{d}s\right\| \\
&\leq \int_t^T \|S_\varepsilon(t-s)\|\,.\,\|B(\varepsilon,s)\|\,.\,\|f_\varepsilon(s,w(s)) - f_\varepsilon(s,v(s))\|\,\mathrm{d}s \\
&\leq k_\varepsilon\int_t^T \varepsilon^{\frac{t-s}{T}}\,\|w - v\|\,\mathrm{d}s \leq \frac{k_\varepsilon}{\varepsilon}\int_t^T \|w - v\|\,\mathrm{d}s \\
&\leq \frac{k_\varepsilon}{\varepsilon}(T-t)|||w(s) - v(s)|||.
\end{aligned}$$

Suppose that (10) holds for $m = j$. We prove that (10) holds for $m = j+1$. Infact, we have

$$\begin{aligned} \left\|F^{j+1}(w)(t) - F^{j+1}(v)(t)\right\| &= \left\|F(F^j(w))(t) - F(F^j(v))(t)\right\| \\ &\le \frac{k_\varepsilon}{\varepsilon}\int_t^T \left\|F^j(w)(s) - F^j(v)(s)\right\| \mathrm{d}s \\ &\le \frac{k_\varepsilon}{\varepsilon}\int_t^T \left(\frac{k_\varepsilon}{\varepsilon}\right)^j \frac{(T-s)^j}{j!} |||w(s) - v(s)||| \,\mathrm{d}s \\ &= \left(\frac{k_\varepsilon}{\varepsilon}\right)^{j+1} \frac{(T-t)^{j+1}}{(j+1)!} |||w(s) - v(s)|||. \end{aligned}$$

Therefore (11) holds for all $m = 1, 2, ...$ by the induction principle. In particular, one has

$$|||F^m(w)(t) - F^m(v)(t)||| \le \left(\frac{k_\varepsilon T}{\varepsilon}\right)^m \frac{1}{m!} |||w(s) - v(s)|||.$$

Since

$$\lim_{m\to\infty} \left(\frac{k_\varepsilon T}{\varepsilon}\right)^m \frac{1}{m!} = 0,$$

there exists a positive integer number m_0 such that F^{m_0} is a contraction mapping. It follows that F^{m_0} has a unique fixed point u^ε in $C([0,T];H)$. Since $F^{m_0}(F(u^\varepsilon)) = F(F^{m_0}(u^\varepsilon)) = F(u^\varepsilon)$, we obtain $F(u^\varepsilon) = u^\varepsilon$ due to the uniqueness of the fixed point of F^{m_0}. The uniqueness of the fixed point of F also follows the uniqueness fixed point of F^{m_0}. The unique fixed point u^ε of F is the solution of (7) corresponding to final value φ.

Step 2: Continuous dependence on the data

We now let u_1^ε and u_2^ε be two solutions corresponding to final values φ_1 and φ_2, respectively. In the same manner as Step 1, we have for every $w, v \in C([0,T];H)$

$$||F(w)(t) - F(v)(t)|| \le k_\varepsilon \int_t^T \varepsilon^{\frac{t-s}{T}} ||w(s) - v(s)|| \,\mathrm{d}s.$$

Hence

$$\begin{aligned} \left\|u_1^\varepsilon(t) - u_2^\varepsilon(t)\right\| &= \left\|S_\varepsilon(t-T)(\varphi_1 - \varphi_2) + F(u_1^\varepsilon)(t) - F(u_2^\varepsilon)(t)\right\| \\ &\le ||S_\varepsilon(t-T)|| \,.\, ||\varphi_1 - \varphi_2|| + \left\|F(u_1^\varepsilon)(t) - F(u_2^\varepsilon)(t)\right\| \\ &\le \varepsilon^{\frac{t-T}{T}} ||\varphi_1 - \varphi_2|| + k_\varepsilon \int_t^T e^{\frac{t-s}{T}} \left\|u_1^\varepsilon(s) - u_2^\varepsilon(s)\right\| ds. \end{aligned}$$

The latter inequality can be written as

$$\varepsilon^{-\frac{t}{T}} \left\|u_1^\varepsilon(t) - u_2^\varepsilon(t)\right\| \le \varepsilon^{-1} ||\varphi_1 - \varphi_2|| + k_\varepsilon \int_t^T e^{-\frac{s}{T}} \left\|u_1^\varepsilon(s) - u_2^\varepsilon(s)\right\| ds.$$

It follows from Gronwall's inequality that

$$\varepsilon^{-\frac{t}{T}} \left\|u_1^\varepsilon(t) - u_2^\varepsilon(t)\right\| \le \varepsilon^{-1} e^{k_\varepsilon(T-t)} ||\varphi_1 - \varphi_2||, \; t \in [0,T].$$

This completes the proof of Theorem 1. □

2.2. Regularization of problem (3)

Our purpose in this section is to construct a regularized solution of the ill-posed problem (3). We mention that the existence of a solution of (3) is not considered here. Instead, we assume that there is an exact solution u corresponding to the exact datum φ, and our aim is to construct, from the given datum φ_ε approximating φ, a regularized solution U_ε which approximates u.

Denote by u^ε the solution of problem (4) corresponding to the final condition φ_ε. We shall show that for each fixed time $t > 0$, the function $u^\varepsilon(t)$ gives a good approximation of $u(t)$, where the order of approximation is $\varepsilon^{\frac{t}{2T}}$. However, it is difficult to derive an approximation at $t = 0$. We therefore need an adjustment in choosing the regularized solution. The main idea is that we first use the continuity of u to approximate the initial value $u(0)$ by $u(t_\varepsilon)$ for some suitable small time $t_\varepsilon > 0$, and then approximate $u(t_\varepsilon)$ by $u^\varepsilon(t_\varepsilon)$. The parameter t_ε will be choosen as follows.

Lemma 3. *Let $T > 0$ and let $\varepsilon > 0$ small enough. There exists a unique $t_\varepsilon > 0$ such that $\varepsilon^{\frac{t_\varepsilon}{2T}} = t_\varepsilon$. Moreover,*

$$t_\varepsilon \le \frac{2T \ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})}.$$

Proof. Note that each solution $t > 0$ of $\varepsilon^{\frac{t}{2T}} = t$ is a zero of the function

$$h(t) = \ln(t) + \frac{\ln(\varepsilon^{-1})}{2T} t, \qquad t > 0.$$

We have h is strictly increasing as $h'(t) > 0$. Moreover, $\lim_{t \to 0^+} h(t) = -\infty$ and

$$h\left(\frac{2T \ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})}\right) = \ln\left[2T \ln(\ln(\varepsilon^{-1}))\right] > 0$$

for $\varepsilon > 0$ small enough. Thus the equation $h(t) = 0$ has a unique solution $t_\varepsilon > 0$ such that

$$t_\varepsilon \le \frac{2T \ln\left(\ln\left(\frac{1}{\varepsilon}\right)\right)}{\ln\left(\frac{1}{\varepsilon}\right)}.$$

□

We have the following regularization result.

Theorem 2. *Let $u \in C^1([0, T]; H)$ be a solution of problem (3) corresponding to $\varphi \in H$. Assume that*

$$\sup_{t \in [0,T]} \left[\sum_{n=1}^{\infty} e^{2T\lambda_n} |(\phi_n, u(t))|^2 + \|u'(t)\| \right] = M < \infty.$$

Let φ_ε be a measured datum satisfying $\|\varphi_\varepsilon - \varphi\| \le \varepsilon$ with $\varepsilon > 0$, and let u^ε be the solution of problem (4) corresponding to φ_ε. Choose $t_\varepsilon > 0$ as in Lemma 3. Define the regularized solution $U^\varepsilon : [0, T] \to H$ by

$$U^\varepsilon(t) = u^\varepsilon(\max\{t, t_\varepsilon\}), \ t \in [0, T].$$

Then one has the error estimate, for $\varepsilon > 0$ small enough, $t \in [0, T]$,

$$\|U^\varepsilon(t) - u(t)\| \le (2M + 1) \min\left\{ \varepsilon^{\frac{t}{2T}}, \frac{2T \ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})} \right\}.$$

Proof. We have in view of (5)

$$u(t) = S(t-T)\varphi - \int_t^T S(t-s)f(s,u(s))\,\mathrm{d}s.$$

Using $B(\varepsilon,t) = S_\varepsilon(t-T)S(T-t)$, one has

$$B(\varepsilon,t)u(t) = S_\varepsilon(t-T)\varphi - \int_t^T S_\varepsilon(t-s)B(\varepsilon,s)f(s,u(s))\,\mathrm{d}s.$$

We have in view of (7)

$$u^\varepsilon(t) = S_\varepsilon(t-T)\varphi_\varepsilon - \int_t^T S_\varepsilon(t-s)B(\varepsilon,s)f_\varepsilon(s,u^\varepsilon(s))\,\mathrm{d}s.$$

Thus

$$\begin{aligned} u^\varepsilon(t)-u(t) &= S_\varepsilon(t-T)\,(\varphi_\varepsilon-\varphi) + [B(\varepsilon,t)-I]\,u(t) + \\ &\quad - \int_t^T S_\varepsilon(t-s)B(\varepsilon,s)\left[f_\varepsilon(s,u^\varepsilon(s)) - f(s,u(s))\right]\,\mathrm{d}s. \end{aligned}$$

Using Lemma 1 and noting that $f(s,u(s)) = f_\varepsilon(s,u(s))$ for $\varepsilon>0$ small enough, $M_\varepsilon \geq \sup_{t\in[0,T]} \|u(t)\|$, we get

$$\begin{aligned} \|u^\varepsilon(t)-u(t)\| &\leq \|S_\varepsilon(t-T)\|\,.\,\|\varphi_\varepsilon-\varphi\| + \|[B(\varepsilon,t)-I]\,u(t)\| + \\ &\quad + \int_t^T \|S_\varepsilon(t-s)\|\,.\,\|B(\varepsilon,s)\|\,.\,\|f_\varepsilon(s,u^\varepsilon(s)) - f(s,u(s))\|\,\mathrm{d}s \\ &\leq \varepsilon^{\frac{t-T}{T}}\cdot\varepsilon + \varepsilon\sqrt{\sum_{n=1}^{\infty} e^{2T\lambda_n}\,|(\phi_n,u)|^2} + k_\varepsilon\int_t^T \varepsilon^{\frac{t-s}{T}}\,\|u^\varepsilon(s)-u(s)\|\,\mathrm{d}s \\ &\leq (M+1)\varepsilon^{\frac{t}{T}} + k_\varepsilon\int_t^T \varepsilon^{\frac{t-s}{T}}\,\|u^\varepsilon(s)-u(s)\|\,\mathrm{d}s. \end{aligned}$$

The latter inequality can be written as

$$\varepsilon^{-\frac{t}{T}}\,\|u^\varepsilon(t)-u(t)\| \leq (M+1) + k_\varepsilon\int_t^T \varepsilon^{-\frac{s}{T}}\,\|u^\varepsilon(s)-u(s)\|\,\mathrm{d}s.$$

It follows from Gronwall's inequality that

$$\varepsilon^{-\frac{t}{T}}\,\|u^\varepsilon(t)-u(t)\| \leq (M+1)e^{k_\varepsilon T},\ \ \forall t\in(0,T].$$

In particular, if $t \in [t_\varepsilon, T]$ then

$$\begin{aligned}\|U^\varepsilon(t) - u(t)\| = \|u^\varepsilon(t) - u(t)\| &\le (M+1)e^{k_\varepsilon T}\varepsilon^{\frac{t}{T}} \\ &\le (M+1)\varepsilon^{\frac{t}{2T}} \le \frac{2T(M+1)\ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})},\end{aligned}$$

where we have used

$$e^{k_\varepsilon T} \le \sqrt{\ln(\varepsilon^{-1})} \le \frac{\ln(\varepsilon^{-1})}{2T\ln(\ln(\varepsilon^{-1}))} \le t_\varepsilon^{-1} = \varepsilon^{-\frac{t_\varepsilon}{2T}} \le \varepsilon^{-\frac{t}{2T}}. \tag{11}$$

Let us now consider $t \in [0, t_\varepsilon]$. One has

$$\|U^\varepsilon(t) - u(t)\| = \|u^\varepsilon(t_\varepsilon) - u(t)\| \le \|u^\varepsilon(t_\varepsilon) - u(t_\varepsilon)\| + \|u(t_\varepsilon) - u(t)\|.$$

Due to the continuity of u_t, we get for ε small enough

$$\|u(t_\varepsilon) - u(t)\| = \left\| \int_t^{t_\varepsilon} u_t(s)ds \right\| \le \int_0^{t_\varepsilon} \|u_t(s)\|\, ds \le Mt_\varepsilon.$$

Thus, for $t \in [0, t_\varepsilon]$,

$$\begin{aligned}\|U^\varepsilon(t) - u(t)\| &\le (M+1)\varepsilon^{\frac{t_\varepsilon}{2T}} + Mt_\varepsilon = (2M+1)t_\varepsilon \\ &\le (2M+1)\min\left\{\varepsilon^{\frac{t}{2T}}, \frac{2T\ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})}\right\}.\end{aligned}$$

This completes the proof of Theorem 2. □

3. Regularization of backward parabolic problem with time-dependent coefficient

In this section, we consider the following backward nonlinear parabolic problem with time-dependent coefficient

$$\begin{aligned} u_t + a(t)Au(t) &= f(t, u(t)), \quad 0 < t < T, \\ u(T) &= \varphi, \end{aligned} \tag{12}$$

where $a \in C([0, T])$ is given. The function a is noised by the perturbed data $a_\varepsilon \in C[0, T]$ such that

$$\|a_\varepsilon - a\|_{C([0,T])} \le \varepsilon. \tag{13}$$

where the norm $\|\cdot\|_{C([0,T])}$ is given by the sup norm, i.e., $\|v\|_{C([0,T])} = \sup_{0 \le t \le T} |v(t)|$ for every continuous function $v : [0, T] \to \mathbb{R}$. We would like to emphasize that it is impossible to apply the technique in Section 2 to solve problem (12) when the time-dependent coefficient is perturbed by noise. Therefore, we investigate a new regularized problem as follows

$$\begin{cases} \dfrac{d}{dt}v_\varepsilon(t) + a_\varepsilon(t)\widetilde{A}_\varepsilon v_\varepsilon(t) = f_\varepsilon(t, v_\varepsilon(t)), & 0 < t < 1, \\ v_\varepsilon(T) = \varphi_\varepsilon, \end{cases} \tag{14}$$

where $\widetilde{A}_\varepsilon$ is defined by

$$\widetilde{A}_\varepsilon(v) := -\frac{1}{QT}\sum_{n=1}^{\infty} \ln\left(\varepsilon + e^{-QT\lambda_n}\right)\langle v, \phi_n\rangle\phi_n \tag{15}$$

and $Q = \|a_\varepsilon\|_{C([0,T])}$.

The regularization result for time-dependent perturbed coefficient is given in the following theorem.

Theorem 3. *Let $u \in C^1([0,T];H)$ be a solution of problem (12) corresponding to $\varphi \in H$. Assume that*

$$\sup_{t\in[0,T]}\left[\sum_{n=1}^{\infty} e^{2QT\lambda_n}|(\phi_n,u(t))|^2 + \|u'(t)\|\right] = E_Q < \infty.$$

Let φ_ε and a_ε be measured data satisfying $\|\varphi_\varepsilon - \varphi\| \le \varepsilon$ and $\|a_\varepsilon - a\|_{C([0,T])} \le \varepsilon$ for $\varepsilon > 0$. We denote by v_ε the solution of problem (14) corresponding to φ_ε and a_ε. Choose $t_\varepsilon > 0$ as in Lemma 3. Define the regularized solution $W^\varepsilon : [0,T] \to H$ by

$$W^\varepsilon(t) = v_\varepsilon(\max\{t,t_\varepsilon\}),\ t \in [0,T].$$

Then one has the following error estimate for $\varepsilon > 0$ small enough and $t \in [0,T]$,

$$\|W^\varepsilon(t) - u(t)\| \le 2E_Q\sqrt{2\left(\frac{1}{Q}+1\right)}e^{2T}\min\left\{\varepsilon^{\frac{t}{2T}}, \frac{2T\ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})}\right\}.$$

Proof. The existence of solutions to problem (12) can be proved in the same manner as Theorem 1. It remains to prove the error estimation between W_ε and u. To this end, we first need the error estimation between u_ε and u. The technique we use here is different from Theorem 2. The problem (12) can be written as

$$\begin{cases} u'(t) + a_\varepsilon(t)\widetilde{A}_\varepsilon u(t) = a_\varepsilon(t)\widetilde{A}_\varepsilon u(t) - a(t)Au(t) + f(t,u(t)), \\ u(T) = \varphi. \end{cases} \tag{16}$$

Recall that v_ε solves the following equation

$$\begin{cases} v_\varepsilon'(t) + a_\varepsilon(t)\widetilde{A}_\varepsilon v_\varepsilon(t) = f_\varepsilon(t,v_\varepsilon(t)), \\ v_\varepsilon(T) = \varphi_\varepsilon. \end{cases} \tag{17}$$

Substituting (17) into (16) bothsides, we obtain

$$\begin{cases} v_\varepsilon'(t) - u'(t) = -a_\varepsilon(t)\widetilde{A}_\varepsilon(v_\varepsilon(t) - u(t)) - a_\varepsilon(t)\widetilde{A}_\varepsilon u(t) + a(t)Au(t) \\ \qquad\qquad +f_\varepsilon(t,v_\varepsilon(t)) - f(t,u(t)), \\ v_\varepsilon(T) - u_\varepsilon(T) = \varphi_\varepsilon - \varphi. \end{cases} \tag{18}$$

For $\widetilde{b} > 0$, we define by

$$z_\varepsilon(t) := e^{\widetilde{b}(t-T)}\Big(v_\varepsilon(t) - u(t)\Big).$$

By differentiating $z_\varepsilon(t)$ with respect t and combining to (18) gives

$$\begin{aligned} z_\varepsilon'(t) &= \widetilde{b}e^{\widetilde{b}(t-T)}\Big(v_\varepsilon(t) - u(t)\Big) + e^{\widetilde{b}(t-T)}(v_\varepsilon'(t) - u'(t)) \\ &= \widetilde{b}z_\varepsilon(t) + e^{\widetilde{b}(t-T)}\Big[-a_\varepsilon(t)\widetilde{A}_\varepsilon\Big(v_\varepsilon(t) - u(t)\Big) + f(t,v_\varepsilon(t)\Big) - f(t,u(t))\Big] \\ &\quad -e^{\widetilde{b}(t-T)}\Bigg[(a_\varepsilon(t) - a(t))Au(t) + a_\varepsilon(t)\Big(\widetilde{A}_\varepsilon - A\Big)u(t)\Bigg] \\ &= \widetilde{b}z_\varepsilon(t) - \widetilde{A}_\varepsilon z_\varepsilon(t) + e^{\widetilde{b}(t-T)}\Big[f(t,v_\varepsilon(t)) - f(t,u(t))\Big] \\ &\quad -e^{\widetilde{b}(t-T)}(a_\varepsilon(t) - a(t))Au(t) - e^{\widetilde{b}(t-T)}a_\varepsilon(t)(\widetilde{A}_\varepsilon - A)u(t). \end{aligned} \tag{19}$$

By taking the inner product (19) with $z_\varepsilon(t)$, we get

$$\begin{aligned}\langle z_\varepsilon'(t) + a_\varepsilon(t)\widetilde{A}_\varepsilon z_\varepsilon(t) - \widetilde{b} z_\varepsilon(t), z_\varepsilon(t)\rangle &= \Big\langle e^{\widetilde{b}(t-T)}\Big[f(t, v_\varepsilon(t)) - f(t, u(t))\Big], z_\varepsilon(t)\Big\rangle \\ &\quad - e^{\widetilde{b}(t-T)}\langle (a_\varepsilon(t) - a(t))Au(t), z_\varepsilon(t)\rangle \\ &\quad - e^{\widetilde{b}(t-T)}\langle (\widetilde{A}_\varepsilon - A)u(t), z_\varepsilon(t)\rangle. \qquad (20)\end{aligned}$$

A direct computation implies that

$$\begin{aligned}\frac{d}{dt}\big\|z_\varepsilon(t)\big\|_H^2 &= 2\langle -a_\varepsilon(t)\widetilde{A}_\varepsilon z_\varepsilon(t), z_\varepsilon(t)\rangle + 2\widetilde{b}\langle z_\varepsilon(t), z_\varepsilon(t)\rangle \\ &\quad + 2\Big\langle e^{\widetilde{b}(t-T)}\Big[f(t, v_\varepsilon(t)) - f(t, u(t))\Big], z_\varepsilon(t)\Big\rangle \\ &\quad - 2e^{\widetilde{b}(t-T)}\langle (a_\varepsilon(t) - a(t))Au(t), z_\varepsilon(t)\rangle \\ &\quad - 2e^{\widetilde{b}(t-T)}\langle \Big(\widetilde{A}_\varepsilon - A\Big)u(t), z_\varepsilon(t)\rangle \\ &= 2(\widetilde{I}_1 + \widetilde{I}_2 + \widetilde{I}_3 + \widetilde{I}_4), \qquad (21)\end{aligned}$$

where

$$\begin{aligned}\widetilde{I}_1 &= \langle -a_\varepsilon(t)\widetilde{A}_\varepsilon z_\varepsilon(t), z_\varepsilon(t)\rangle + \widetilde{b}\langle z_\varepsilon(t), z_\varepsilon(t)\rangle, \\ \widetilde{I}_2 &= \Big\langle e^{\widetilde{b}(t-T)}\Big[f_\varepsilon(t, v_\varepsilon(t)) - f(t, u(t))\Big], z_\varepsilon(t)\Big\rangle, \\ \widetilde{I}_3 &= -e^{\widetilde{b}(t-T)}\langle (a_\varepsilon(t) - a(t))Au(t), z_\varepsilon(t)\rangle, \\ \widetilde{I}_4 &= -e^{\widetilde{b}(t-T)}\Big\langle \Big(\widetilde{A}_\varepsilon - A\Big)u(t), z_\varepsilon(t)\Big\rangle.\end{aligned}$$

Since $Q = \sup_{t\in[0,T]} \big|a_\varepsilon(t)\big|$, we have

$$\begin{aligned}\Big|\langle -a_\varepsilon(t)\widetilde{A}_\varepsilon z_\varepsilon(t), z_\varepsilon(t)\rangle\Big| &\le \sup_{t\in[0,1]} \big|a_\varepsilon(t)\big|\big\|\widetilde{A}_\varepsilon z_\varepsilon(t)\big\|_H\big\|z_\varepsilon(t)\big\|_H \\ &\le Q\frac{1}{QT}\ln\left(\frac{1}{\varepsilon}\right)\big\|z_\varepsilon(t)\big\|_H^2 \\ &\le \frac{1}{T}\ln\left(\frac{1}{\varepsilon}\right)\big\|z_\varepsilon(t)\big\|_H^2,\end{aligned}$$

which gives

$$\langle -a_\varepsilon(t)\widetilde{A}_\varepsilon z_\varepsilon(t), z_\varepsilon(t)\rangle \ge -\frac{1}{T}\ln\left(\frac{1}{\varepsilon}\right)\big\|z_\varepsilon(t)\big\|_H^2.$$

Then the term $\widetilde{I}_1$ is estimated by

$$\begin{aligned}\widetilde{I}_1 &= \langle -a_\varepsilon(t)\widetilde{A}_\varepsilon z_\varepsilon(t), z_\varepsilon(t)\rangle + \widetilde{b}\langle z_\varepsilon(t), z_\varepsilon(t)\rangle \\ &\ge -\frac{1}{T}\ln\left(\frac{1}{\varepsilon}\right)\big\|z_\varepsilon(t)\big\|_H^2 + \widetilde{b}\big\|z_\varepsilon(t)\big\|_H^2. \qquad (22)\end{aligned}$$

Using Lemma 1 and noting that $f(s, u(s)) = f_\varepsilon(s, u(s))$ for $\varepsilon > 0$ small enough, $M_\varepsilon \ge \sup_{t\in[0,T]} \|u(t)\|$, we have the following estimate

$$\begin{aligned}\widetilde{I}_2 &= \Big\langle e^{-\widetilde{b}(T-t)}\Big[f_\varepsilon(t, v_\varepsilon(t)) - f(t, u(t))\Big], z_\varepsilon(t)\Big\rangle \\ &= e^{-2\widetilde{b}(T-t)}\langle f_\varepsilon(v_\varepsilon(t), t) - f_\varepsilon(t, u(t)), v_\varepsilon(t) - u(t)\rangle \\ &\ge -k_\varepsilon e^{-2\widetilde{b}(T-t)}\big\|v_\varepsilon(t) - u(t)\big\|_H^2 \\ &= -k_\varepsilon\big\|z_\varepsilon\big\|_H^2. \qquad (23)\end{aligned}$$

Employing Hölder inequality, we can bound $\widetilde{I_3}$ as follows

$$\begin{aligned}
\widetilde{I_3} &= \langle e^{-\widetilde{b}(T-t)}(a_\varepsilon(t)-a(t))Au(t), z_\varepsilon(t)\rangle \\
&\le e^{-2\widetilde{b}(T-t)}\left|a_\varepsilon(t)-a(t)\right|^2\left\|Au(t)\right\|_H^2 + \left\|z_\varepsilon(t)\right\|_H^2 \\
&\le e^{-2\widetilde{b}(T-t)}\left|a_\varepsilon(t)-a(t)\right|^2\left(\sum_{n=1}^{\infty}\lambda_n^2\left|\langle u(t),\phi_n\rangle\right|^2\right) + \left\|z_\varepsilon(t)\right\|_H^2 \\
&\le e^{-2\widetilde{b}(T-t)}\left|a_\varepsilon(t)-a(t)\right|^2\left(\sum_{n=1}^{\infty}\frac{1}{Q^2T^2}\, e^{2QT\lambda_n}\left|\langle u(t),\phi_n\rangle\right|^2\right) + \left\|z_\varepsilon(t)\right\|_H^2 \\
&\le \frac{e^{-2\widetilde{b}(T-t)}\varepsilon^2E_Q^2}{QT} + \left\|z_\varepsilon(t)\right\|_H^2. \qquad (24)
\end{aligned}$$

Using Hölder inequality again, $\widetilde{I_4}$ can be bounded as

$$\begin{aligned}
\widetilde{I_4} &= \langle e^{-\widetilde{b}(T-t)}a_\varepsilon(t)(\widetilde{A}_\varepsilon(t)-A(t))u(t), z_\varepsilon(t)\rangle \\
&\le e^{-2\widetilde{b}(T-t)}\left|a_\varepsilon(t)\right|^2\left\|(\widetilde{A}_\varepsilon - A)u(t)\right\|_H^2 + \left\|z_\varepsilon(t)\right\|_H^2 \\
&\le e^{-2\widetilde{b}(T-t)}\left|a_\varepsilon(t)\right|^2\sum_{n=1}^{\infty}\left|\frac{1}{QT}\ln\left(\frac{1}{\varepsilon+e^{-QT\lambda_n}}\right)-\frac{1}{QT}\ln(e^{QT\lambda_n})\right|^2\left|\langle u(t),\phi_n\rangle\right|^2 \\
&\quad +\left\|z_\varepsilon(t)\right\|_H^2 \\
&\le e^{-2\widetilde{b}(T-t)}\left|a_\varepsilon(t)\right|^2\frac{1}{Q^2T^2}\sum_{n=1}^{\infty}\left|\ln\left(\frac{1}{\varepsilon e^{QT\lambda_n}+1}\right)\right|^2\left|\langle u(t),\phi_n\rangle\right|^2 + \left\|z_\varepsilon(t)\right\|_H^2 \\
&\le \frac{1}{T^2}e^{-2\widetilde{b}(T-t)}\sum_{n=1}^{\infty}\ln^2\left(\varepsilon e^{QT\lambda_n}+1\right)\left|\langle u(t),\phi_n\rangle\right|^2 + \left\|z_\varepsilon(t)\right\|_H^2 \\
&\le \frac{1}{T^2}e^{-2\widetilde{b}(T-t)}\varepsilon^2\sum_{n=1}^{\infty}e^{2QT\lambda_n}\left|\langle u(t),\phi_n\rangle\right|^2 + \left\|z_\varepsilon(t)\right\|_H^2 \\
&\le \frac{1}{T^2}e^{-2\widetilde{b}(T-t)}\varepsilon^2E_Q^2 + \left\|z_\varepsilon(t)\right\|_H^2. \qquad (25)
\end{aligned}$$

Thus, (21), (22), (23), (24) and (25) yields

$$\begin{aligned}
\frac{d}{dt}\left\|z_\varepsilon(t)\right\|_H^2 &\ge \left(-\frac{2}{T}\ln\left(\frac{1}{\varepsilon}\right)+2\widetilde{b}-2k_\varepsilon-4\right)\left\|z_\varepsilon(t)\right\|_H^2 \\
&\quad -2e^{-2\widetilde{b}(T-t)}\varepsilon^2E_Q^2\left(\frac{1}{QT}+\frac{1}{T}\right). \qquad (26)
\end{aligned}$$

Since $b = \frac{1}{T}\ln\left(\frac{1}{\varepsilon}\right)$ we obtain

$$\frac{d}{dt}\left\|z_\varepsilon(t)\right\|_H^2 \ge (-2k_\varepsilon-4)\left\|z_\varepsilon(t)\right\|_H^2 - 2\varepsilon^2E_Q^2\left(\frac{1}{QT}+\frac{1}{T}\right).$$

Integrating the above inequality from t to T, we get

$$\begin{aligned}
\left\|z_\varepsilon(T)\right\|_H^2 - \left\|z_\varepsilon(t)\right\|_H^2 &\ge (-2k_\varepsilon-4)\int_t^T\left\|z_\varepsilon(s)\right\|_H^2 ds \\
&\quad -2E_Q^2\varepsilon^2\left(\frac{1}{QT}+\frac{1}{T}\right)(T-t).
\end{aligned}$$

Since $\left\|z_\varepsilon(T)\right\|_H^2 = \|\varphi_\varepsilon - \varphi\| \le \varepsilon$, we have

$$\left\|z_\varepsilon(t)\right\|_H^2 \le (2k_\varepsilon + 4)\int_t^1 \left\|z_\varepsilon(s)\right\|_H^2 ds + 2E_Q^2\varepsilon^2\left(\frac{1}{Q}+1\right)+\varepsilon^2.$$

This implies that

$$\begin{aligned} e^{-2\widetilde{b}(T-t)}\left\|v_\varepsilon(t)-u(t)\right\|_H^2 &\le (2k_\varepsilon+4)\int_t^T e^{-2\widetilde{b}(T-s)}\left\|v_\varepsilon(s)-u(s)\right\|_H^2 ds \\ &\quad +2E_Q^2\varepsilon^2\left(\frac{1}{Q}+1\right)+\varepsilon^2. \end{aligned}$$

Multiplying bothside to $e^{2\widetilde{b}T}$, we obtain

$$\begin{aligned} e^{2\widetilde{b}t}\left\|v_\varepsilon(t)-u(t)\right\|_H^2 &\le (2k_\varepsilon+4)\int_t^T e^{2bs}\left\|v_\varepsilon(s)-u(s)\right\|_H^2 ds \\ &\quad +2E_Q^2\left(\frac{1}{Q}+1\right). \end{aligned}$$

Applying Grönwall's inequality, we get

$$e^{2\widetilde{b}t}\left\|v_\varepsilon(t)-u(t)\right\|_H^2 \le 2E_Q^2\left(\frac{1}{Q}+1\right)e^{\int_t^T(2k_\varepsilon+4)ds},$$

or

$$e^{2\widetilde{b}t}\left\|v_\varepsilon(t)-u(t)\right\|^2 \le 2E_Q^2\left(\frac{1}{Q}+1\right)e^{(2k_\varepsilon+4)(T-t)}.$$

Hence

$$\left\|v_\varepsilon(t)-u(t)\right\|_H^2 \le 2E_Q^2\left(\frac{1}{Q}+1\right)e^{(2k_\varepsilon+4)(T-t)}e^{-\frac{2t}{T}\ln\left(\frac{1}{\varepsilon}\right)}.$$

In particular, if $t\in[t_\varepsilon,T]$ then

$$\begin{aligned} \|W^\varepsilon(t)-u(t)\| = \|v_\varepsilon(t)-u(t)\| &\le E_Q\sqrt{2\left(\frac{1}{Q}+1\right)}e^{2T}e^{k_\varepsilon T}\varepsilon^{\frac{t}{T}} \\ &\le E_Q\sqrt{2\left(\frac{1}{Q}+1\right)}e^{2T}\varepsilon^{\frac{t}{2T}} \\ &\le E_Q\sqrt{2\left(\frac{1}{Q}+1\right)}e^{2T}\frac{2T\ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})}, \end{aligned}$$

where we have used (11).

Let us now consider $t\in[0,t_\varepsilon]$. One has

$$\|W^\varepsilon(t)-u(t)\| = \|v_\varepsilon(t_\varepsilon)-u(t)\| \le \|v_\varepsilon(t_\varepsilon)-u(t_\varepsilon)\| + \|u(t_\varepsilon)-u(t)\|.$$

Due to the continuity, we get for ε small enough

$$\|u(t_\varepsilon) - u(t)\| = \left\| \int_t^{t_\varepsilon} u_t(s)ds \right\| \le \int_0^{t_\varepsilon} \|u_t(s)\| \, ds \le E_Q t_\varepsilon.$$

Thus, for $t \in [0, t_\varepsilon]$,

$$\begin{aligned} \|W^\varepsilon(t) - u(t)\| &\le E_Q \sqrt{2\left(\frac{1}{Q}+1\right)} e^{2T} \varepsilon^{\frac{t_\varepsilon}{2T}} + E_Q t_\varepsilon \\ &\le 2E_Q \sqrt{2\left(\frac{1}{Q}+1\right)} e^{2T} \min\left\{ \varepsilon^{\frac{t}{2T}}, \frac{2T \ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})} \right\}. \end{aligned}$$

This completes the proof of Theorem 3. □

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