

Recovering a Class of Entire Functions and Application to Heat Equations

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Abstract. We consider the problem of recovering a class of entire functions from their values on a set of integer nodes. These results are applied to investigate two problems involving heat equations: the first one is of solving a heat equation without either the initial condition or the final condition, and the second one is of determining the heat source of a backward heat problem.

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1. Introduction

For $\sigma > 0$, we denote by L_σ^2 the space of entire functions $f \in L^2(\mathbb{R})$ satisfying

$$|f(z)| \leq \text{const} \cdot e^{\sigma|z|}, \quad z \in \mathbb{C}.$$

By Paley-Wiener theorem (see Rudin [10, Chapter 19]), each function $f \in L_\sigma^2$ can be represented as the Fourier transform of a function $g \in L^2(-\sigma, \sigma)$, i.e.

$$f(z) = \int_{-\sigma}^{\sigma} g(t)e^{itz} dt, \quad z \in \mathbb{C}.$$

We are interested in a problem of recovering a function in L^2_σ from its values on a certain subset of \mathbb{R} , which is known when considering some partial differential equations.

First, let us consider the heat equation

$$\begin{cases} u_t - u_{xx} = f(x, t), & (x, t) \in Q := (0, 1) \times (0, T), \\ u_x(0, t) = u_x(1, t) = u(1, t) = 0, \end{cases} \quad (1)$$

where $u \in C^1([0, T]; L^1(0, 1)) \cap L^2(0, T; H^2(0, 1))$ is unknown. Here we recall that $C^1([0, T]; L^1(0, 1))$ is the space of all continuous functions $f : [0, T] \rightarrow L^1(0, 1)$ having $f' : [0, T] \rightarrow L^1(0, 1)$ continuous and that $L^2(0, T; H^2(0, 1))$ is the space of all $f : (0, T) \rightarrow H^2(0, 1)$ satisfying

$$\int_0^T \|f(t)\|_{H^2(0,1)}^2 < \infty.$$

This is a kind of the problems called “problems without initial conditions”. In 1935, Tikhonov [16] proved the uniqueness of the solution of the homogeneous heat equation

$$u_t - \Delta u = 0, \quad -\infty < t < \infty.$$

In 1990, Safarov [11] solved this homogeneous problem for the unbounded domain $x > 0$ and for the trip $0 < x < l$. After that, the nonhomogeneous heat equation without initial condition has been considered by many authors such as Shmulev [15], Kirilich [8] and Guseinov [7]. These authors investigated the problem in $-\infty < t < \infty$ or $-\infty < t < T$ and they required some assumptions on the behavior of the temperature at $-\infty$ or a periodic condition in order to obtain the solvability of the problem. In the present paper, we shall consider the nonhomogeneous problem on a finite time $0 < t < T$, which is more reasonable for real applications. The lack of the initial condition $u(\cdot, 0)$ is compensated by adding the boundary condition $u(1, \cdot)$. We want to prove that the problem (1) has at most one solution and solve it numerically.

For each $\alpha \in \mathbb{R}$, getting the inner product between the first equation of (1) and $v(x) = \cos(\alpha x)$, and using the integration by parts, one has

$$\frac{d}{dt} F(u(\cdot, t))(\alpha) + \alpha^2 F(u(\cdot, t))(\alpha) = F(f(\cdot, t))(\alpha),$$

where F stands for the Fourier cosin transform in $L^2(0, 1)$, i.e.

$$F(w)(z) := \int_0^1 w(x) \cos(zx) dx, \quad w \in L^2(0, 1), \quad z \in \mathbb{C}.$$

It follows from the above equation that

$$F(u(\cdot, T))(\alpha) = e^{-\alpha^2 T} F(u(\cdot, 0))(\alpha) + \int_0^T e^{\alpha^2(t-T)} F(f(\cdot, t))(\alpha) dt. \quad (2)$$

The difficulty of finding $u(\cdot, T)$ is because the initial condition $u(\cdot, 0)$ is not available. However, from (2) we have a crucial observation that when $|\alpha| \rightarrow +\infty$ then $e^{-\alpha^2 T} \rightarrow 0$ very fast and it is reasonable to use the approximation

$$F(u(\cdot, T))(\alpha) \approx \int_0^T e^{\alpha^2(t-T)} F(f(\cdot, t))(\alpha) dt. \quad (3)$$

Formula (3) gives a good approximation for $F(u(\cdot, T))(\alpha)$ if $|\alpha|$ is large enough. Now the crux of the matter is to recover $F(u(\cdot, T))(\alpha)$ for $|\alpha|$ small as well. Thus we meet a problem of recovering a function in L_1^2 from its values on a set $(-\infty, -r] \cup [r, \infty)$, where $r > 0$ is a large number.

Now let us consider another heat problem called “inverse source problem”. This is of finding a pair of function (u, f) satisfying

$$\begin{cases} u_t - u_{xx} = \varphi(t)f(x), & (x, t) \in Q = (0, 1) \times (0, T) \\ u_x(0, t) = u_x(1, t) = u(1, t) = 0, \\ u(x, T) = g(x), \end{cases} \quad (4)$$

where $\varphi \in L^1(0, T)$ and $g \in L^2(0, 1)$ are given. This inverse source problem is ill-posed, i.e. a solution may not exist, and even if it exists then it may not depend continuously on the data. Therefore, an usual numerical treatment is impossible and a regularization is necessary.

The problem of finding the heat source under separate form $\varphi(t)f(x)$, where one of the two functions φ and f is unknown, has been investigated for a long time. The uniqueness and stability are considered by many authors such as Cannon-Esteva [3, 4], Yamamoto [21, 22], Yamamoto-Zou [23], Saitoh-Tuan-Yamamoto [12, 13] and Choulli-Yamamoto [5]. However, the regularization problem for unstable cases is still difficult. The regularization problem for the case $f \equiv 1$ was investigated by Wang-Zheng [19] and Shidfar-Zakeri-Neisi [14], while the case $\varphi \equiv 1$ was considered by Cannon [2], Wang-Zheng [20] and Farcas-Lesnic [6]. Recently, Trong-Long-Dinh [17] and Trong-Quan-Dinh [18] regarded the regularization problem in which φ is given and f is unknown. However, in the two latter papers, both of the initial condition $u(\cdot, 0)$ and the final condition $u(\cdot, T)$ are required. This requirement is strict and unnatural. In the present paper, we consider the same problem as in [17] but the requirement of the initial temperature is removed completely.

Note that if f is already known then we obtain an usual backward heat problem. Therefore, we shall focus only on finding of f . From (4), we have the following formula, which comes from (2),

$$F(g)(\alpha) - e^{-\alpha^2 T} F(u(\cdot, 0)) = D(\varphi)(\alpha) F(f)(\alpha), \quad \alpha \in \mathbb{R}, \quad (5)$$

where

$$D(\varphi)(\alpha) = \int_0^T e^{\alpha^2(t-T)} \varphi(t) dt.$$

If $e^{-\alpha^2 T} / |D(\varphi)(\alpha)| \rightarrow 0$ "fast enough" when $|\alpha| \rightarrow +\infty$ then we have the approximation

$$F(f)(\alpha) = \frac{F(g)(\alpha)}{D(\varphi)(\alpha)} - e^{-\alpha^2 T} \frac{F(u(\cdot, 0))(z)}{D(\varphi)(\alpha)} \approx \frac{F(g)(\alpha)}{D(\varphi)(\alpha)}. \quad (6)$$

Thus we meet again the problem of recovering a function of L^2_1 , i.e. $F(f)$, from its values on a set $(-\infty, -r] \cup [r, \infty)$, where $r > 0$ is a large number.

In summary, two heat problems suggest us a "tool problem" of recovering functions in L^2_σ . The remainder of the paper is divided into three sections. In Sec. 2, we give some results about this "tool problem". In Sec. 3, we return to the heat problems and apply the results in Sec. 2 to solve them. A numerical experiment is presented in Sec. 4 to illuminate the effect of our method.

2. Recovering Functions in L^2_σ from Integer Nodes

In this section, we consider only a special case of the "tool problem". More precisely, we consider the problem of recovering a function in L^2_σ from its values on a set $\{n \in \mathbb{Z}, |n| \geq r\}$ for some $r > 0$ large.

The problem of finding an entire function from its value on a countable set A is a typical problem in the theory of entire functions. In particular, the interesting case $A = \mathbb{Z}$ has been considered for a long time (see [9, Lecture 20]). A classical theorem (see [9, Section 20.2]) says that a function $f \in L^2_\pi$ can be determined completely from its values on $A = \mathbb{Z}$ by the representation

$$f(z) = \sum_{n=-\infty}^{\infty} (-1)^n f(n) \frac{\sin(\pi(z-n))}{\pi(z-n)},$$

and

$$\|f\|_{L^2(\mathbb{R})}^2 = \sum_{n \in \mathbb{Z}} |f(n)|^2.$$

However, if $A = \{n \in \mathbb{Z}, |n| \geq r\}$ for some $r > 0$ then the uniqueness of the determination of $f \in L^2_\pi$ does not hold. For example, the entire function

$$f(z) = \frac{\sin(\pi z)}{z}$$

satisfies $f(n) = 0$ for all integers $n \neq 0$. Hence, it is interesting to consider the determination of a function $f \in L^2_\sigma$ from $\{f(n)\}_{n \in A}$ in the case that $\sigma < \pi$.

When $0 < \sigma < 2$, we have the following stability theorem. In particular, it implies the uniqueness result.

Theorem 2.1. (Stability) *Let $\sigma \in (0, 2)$ and $A = \{n \in \mathbb{Z}, |n| > r\}$ for some $r > 0$. Then there exists a constant $C = C(\sigma, r) > 0$ such that*

$$\|f\|_{L^2(\mathbb{R})} \leq C \left(\sum_{n \in A} |f(n)|^2 \right)^{1/2}, \quad \forall f \in L^2_\sigma.$$

Remark 2.2. Since $\|f\|_{L^2(\mathbb{R})}^2 = \sum_{n \in \mathbb{Z}} |f(n)|^2$, the above stability theorem can be said in other words that the mapping

$$f \mapsto \left(\sum_{n \in A} |f(n)|^2 \right)^{1/2}$$

defines an equivalent norm in the space $(L^2_\sigma, \|\cdot\|_{L^2(\mathbb{R})})$.

Our main tool to prove Theorem 2.1 is the **Lagrange interpolation polynomial**. Recall that if $B = \{x_1, \dots, x_p\}$ is a set of p mutually distinct complex numbers then the Lagrange interpolation polynomial $L[B; w]$ of a mapping $w : B \rightarrow \mathbb{C}$ is

$$L[B; w](z) = \sum_{j=1}^p \left(\prod_{k \neq j} \frac{z - x_k}{x_j - x_k} \right) w(x_j).$$

We shall need the following interpolation inequality.

Lemma 2.3. (Interpolation inequality) *Let $\sigma_0 \in (\sigma, 2)$. Then there exist $k \in \mathbb{N}$ and $C_0 > 0$ depending only on σ_0 such that*

$$\sup_{x \in [-r, r]} |f(x) - L[A_r^k, f]| \leq C_0 \|f\|_{L^2(\mathbb{R})} \left(\frac{\sigma}{\sigma_0} \right)^{2kr}, \quad \forall f \in L^2_\sigma, r \in \mathbb{N},$$

where $A_r^k = \{\pm(r+j) | j = 1, 2, \dots, kr\}$.

Proof. Fix $x \in [-r, r]$ and put $x_j = r + j$ for $1 \leq j \leq kr$. According to the remainder formula of the Lagrange interpolation polynomial (see, e.g., [1, Section 1.1, p. 9]), there exists $\xi \in [-(k+1)r, (k+1)r]$ such that

$$f(x) - L[A_r^k, f](x) = \frac{1}{(2kr)!} \frac{d^{2kr} f}{dz^{2kr}}(\xi) \prod_{j=1}^{kr} (x^2 - x_j^2). \quad (7)$$

Since $f \in L^2_\sigma$, we have the representation

$$f(z) = \int_{-\sigma}^{\sigma} g(t)e^{itz} dt$$

for some $g \in L^2(-\sigma, \sigma)$. Thus

$$\left| \frac{d^{2kr} f}{dz^{2kr}}(\xi) \right| = \left| \int_{-\sigma}^{\sigma} g(t)(it)^{2kr} e^{it\xi} dt \right| \leq \|g\|_{L^1(-\sigma, \sigma)} \sigma^{2kr} \leq \|f\|_{L^2(-\sigma, \sigma)} \sigma^{2kr}, \quad (8)$$

here we have used

$$\|g\|_{L^1(-\sigma, \sigma)} \leq \sqrt{2\sigma} \|g\|_{L^2(-\sigma, \sigma)} = \sqrt{\frac{\sigma}{\pi}} \|f\|_{L^2(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})}.$$

On the other hand, since $|x| \leq x_j$,

$$\begin{aligned} \frac{1}{(2kr)!} \prod_{j=1}^{kr} |x^2 - x_j^2| &\leq \frac{1}{(2kr)!} \prod_{j=1}^{kr} x_j^2 \\ &= \frac{(r+1)^2 (r+2)^2 \dots ((k+1)r)^2}{(2kr)!} =: \Psi_k(r). \end{aligned} \quad (9)$$

We see that

$$\frac{\Psi_k(r+1)}{\Psi_k(r)} = \frac{(k+1)^2 ((k+1)r+1)^2 \dots ((k+1)r+k)^2}{(2kr+1) \dots (2kr+2k)} \rightarrow \frac{(k+1)^{2(k+1)}}{(2k)^{2k}}$$

as $r \rightarrow +\infty$.

Because

$$\left(1 + \frac{1}{k}\right) \ln(k+1) - \ln(2k) \rightarrow \ln\left(\frac{1}{2}\right) < -\ln(\sigma_0) \text{ as } k \rightarrow +\infty,$$

we can choose $k > 0$ depending on σ_0 such that

$$\left(1 + \frac{1}{k}\right) \ln(k+1) - \ln(2k) < -\ln(\sigma_0).$$

The latter inequality is equivalent to

$$\frac{(k+1)^{2(k+1)}}{(2k)^{2k}} < \frac{1}{(\sigma_0)^{2k}}.$$

It follows from

$$\frac{\Psi_k(r+1)}{\Psi_k(r)} \rightarrow \frac{(k+1)^{2(k+1)}}{(2k)^{2k}} < \frac{1}{(\sigma_0)^{2k}} \text{ as } r \rightarrow +\infty,$$

that

$$\Psi_k(r) \leq \frac{C_0}{(\sigma_0)^{2kr}}, \quad \forall r = 1, 2, \dots \quad (10)$$

for some constant $C_0 > 0$ depending only on σ_0 . The desired result follows from (7), (8), (9) and (10). \blacksquare

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. We first prove the uniqueness and then obtain the stability.

Step 1. We prove that if $f \in L^2_\sigma$ and $f(n) = 0$ for all $n \in A$ then $f \equiv 0$.

In fact, choose $\sigma_0 \in (\sigma, 2)$ and k, C_0 as in Lemma 2.3. Fix $x \in \mathbb{R}$. Then $A_s^k \subset A$ for $s > |x|$ large enough. It follows from Lemma 2.3 that

$$|f(x)| = |f(x) - L[A_s^k, f](x)| \leq C_0 \|f\|_{L^2(R)} \left(\frac{\sigma}{\sigma_0}\right)^{2ks} \rightarrow 0 \quad \text{as } s \rightarrow +\infty.$$

Thus $f(x) = 0$ for all $x \in \mathbb{R}$, and hence $f \equiv 0$.

Step 2. Assume by contradiction that there is a sequence $\{f_m\} \subset L^2_\sigma$ satisfying

$$\|f_m\|_{L^2(R)} = 1 \quad \text{and} \quad \lim_{m \rightarrow +\infty} \sum_{n \in A} |f_m(n)|^2 = 0.$$

Since $f_m \in L^2_\sigma$, there exists $g_m \in L^2(-\sigma, \sigma)$ such that

$$f_m(z) = \int_{-\sigma}^{\sigma} g_m(t) e^{itz} dt, \quad \|g_m\|_{L^2(-\sigma, \sigma)} = \frac{1}{\sqrt{2\pi}} \|f_m\|_{L^2(R)} = \frac{1}{\sqrt{2\pi}}.$$

It follows from the boundedness of the sequence $\{g_m\}$ in $L^2(-\sigma, \sigma)$ that there is a subsequence $\{g_{m_k}\}$ weakly converging to a function $g_0 \in L^2(-\sigma, \sigma)$. Thus

$$\lim_{k \rightarrow +\infty} f_{m_k}(n) = f_0(n), \quad \forall n \in \mathbb{Z}, \quad \text{where } f_0(z) = \int_{-\sigma}^{\sigma} g_0(t) e^{itz} dt.$$

On the other hand, $\lim_{k \rightarrow +\infty} f_{m_k}(n) = 0$ for all $n \in A$. Therefore, $f_0(n) = 0$ for all $n \in A$, and we deduce from Theorem 2.1 that $f_0 \equiv 0$. Thus $\lim_{k \rightarrow +\infty} f_{m_k}(n) \rightarrow 0$ for all $n \in \mathbb{Z}$, and hence

$$\|f_{m_k}\|_{L^2(R)}^2 = \sum_{n \in \mathbb{Z}} |f_{m_k}(n)|^2 = \sum_{|n| < r} |f_{m_k}(n)|^2 + \sum_{n \in A} |f_{m_k}(n)|^2 \rightarrow 0$$

as $k \rightarrow +\infty$. This contradiction completes the proof. \blacksquare

Now we sharp the interpolation inequality in Lemma 2.3 for the special case $\sigma = 1$.

Theorem 2.4. (Interpolation inequality) *Let $f \in L_1^2$ and $w : A_r \rightarrow \mathbb{C}$, where $A_r = \{\pm(r+j), j = 1, 2, \dots, 4r\}$ for some integer $r > 0$. Then one has*

$$\sup_{x \in [-r, r]} |f(x) - L[A_r; w](x)| \leq \|f\|_{L^2(R)} e^{-r/2} + 8r e^{3.96r} \max_{n \in A_r} |f(n) - w(n)|.$$

Proof. Fix $x \in [-r, r]$ and put $m = 4r$, $x_j = r + j$ for $1 \leq j \leq m$. We shall use the triangle inequality

$$|f(x) - L[A_r; w](x)| \leq |f(x) - L[A_r; f](x)| + |L[A_r; (f - w)](x)|. \quad (11)$$

We first estimate $|f(x) - L[A_r; f](x)|$. Using (7), (8), (9) in the proof of Lemma 2.3 with respect to $k = 4$ we get

$$|f(x) - L[A_r; f](x)| \leq \|f\|_{L^2(R)} \Psi_4(r), \quad (12)$$

where

$$\Psi_4(r) = \frac{(r+1)^2(r+2)^2 \dots (5r)^2}{(8r)!}.$$

By direct computation, one has $\Psi_4(1) = 4/15 < e^{-1/2}$, and

$$\frac{\Psi_4(r+1)}{\Psi_4(r)} = \frac{25 [(5r+1)(5r+2)(5r+3)(5r+4)]^2}{(8r+1)(8r+2)\dots(8r+8)} < \frac{5^{10}}{8^8} < e^{-1/2}$$

since

$$\begin{aligned} & 5^8(8r+1)(8r+2)\dots(8r+8) - 8^8[(5r+1)(5r+2)(5r+3)(5r+4)]^2 \\ &= 327680000000r^7 + 1134592000000r^6 + 1611776000000r^5 \\ & \quad + 12084267520000r^4 + 5110135040000r^3 + 1199880928000r^2 \\ & \quad + 141123408000r + 6086323584 \\ & > 0. \end{aligned}$$

It implies that $\Psi_4(r) < e^{-r/2}$ for all $r \geq 1$. Thus (11) reduces to

$$|f(x) - L[A_r; f](x)| \leq \|f\|_{L^2(R)} e^{-r/2}. \quad (13)$$

We next estimate $|L[A_r; f - w](x)|$. We have

$$\begin{aligned} L[A_r; f - w](x) &= \sum_{j=1}^m \left(\prod_{k \neq j} \frac{x^2 - x_k^2}{x_j^2 - x_k^2} \right) \frac{x + x_j}{2x_j} (f(x_j) - w(x_j)) \\ & \quad + \sum_{j=1}^m \left(\prod_{k \neq j} \frac{x^2 - x_k^2}{x_j^2 - x_k^2} \right) \frac{x - x_j}{-2x_j} (f(-x_j) - w(-x_j)). \end{aligned}$$

Since $|x| \leq x_j$ for $1 \leq j \leq m$, the above equality implies

$$\begin{aligned}
 |L[A_r; f - w](x)| &\leq 2\delta \sum_{j=1}^m \left(\prod_{k \neq j} \frac{|x^2 - x_k^2|}{|x_j^2 - x_k^2|} \right) \\
 &\leq 2\delta m \max_{1 \leq j \leq m} \left(\prod_{k \neq j} \frac{x_k^2}{|x_j^2 - x_k^2|} \right), \tag{14}
 \end{aligned}$$

where $\delta = \max_{n \in A_r} |f(n) - w(n)|$. For all $1 \leq j \leq m$,

$$\begin{aligned}
 \prod_{k \neq j} \frac{x_k^2}{|x_j^2 - x_k^2|} &= \frac{2}{x_j} \cdot \frac{\prod_{k=1}^m x_k^2}{\prod_{k \neq j} |x_j - x_k| \cdot \prod_{k=1}^m (x_j + x_k)} \\
 &= \frac{2}{r+j} \cdot \frac{(r+1)^2 (r+2)^2 \dots (5r)^2}{(j-1)! (4r-j)! (2r+j+1) (2r+j+2) \dots (6r+j)} \\
 &\leq \frac{4(r+1)^2 (r+2)^2 \dots (5r)^2}{(2r-1)! (6r+1)!} =: \tilde{\Psi}(r)
 \end{aligned}$$

since

$$\begin{aligned}
 &2(r+j) \times (j-1)! (4r-j)! \times (2r+j+1) \cdot (2r+j+2) \dots (6r+j) \\
 &\geq (2r+1) \times (2r-1)! (2r)! \times (2r+2) \cdot (2r+3) \dots (6r+1) \\
 &= (2r-1)! (6r+1)!.
 \end{aligned}$$

By direct computation, we have $\tilde{\Psi}(1) < e^{3.96}$, and

$$\frac{\tilde{\Psi}(r+1)}{\tilde{\Psi}(r)} = \frac{25[(5r+1)(5r+2)(5r+3)(5r+4)]^2}{2r(2r+1)(6r+2)(6r+3) \dots (6r+7)} < \frac{5^{10}}{2^2 \cdot 6^6} < e^{3.96}$$

since

$$\begin{aligned}
 &5^8 \cdot 2r(2r+1)(6r+2) \dots (6r+7) - 2^2 \cdot 6^6 \cdot [(5r+1) \dots (5r+4)]^2 \\
 &= 72900000000r^7 + 26568000000r^6 + 39406500000r^5 \\
 &\quad + 302946030000r^4 + 125967060000r^3 + 26004042000r^2 \\
 &\quad + 1698012000r - 107495424 \\
 &> 0.
 \end{aligned}$$

Thus $\tilde{\Psi}(r) < e^{3.96r}$ for all $r \geq 1$. Therefore, (14) reduces to

$$|L[A_r; f - w](x)| \leq 8re^{3.96r} \delta. \tag{15}$$

The desired result follows inequalities (13) and (15). \blacksquare

Corollary 2.5. Assume that $f \in L_1^2$ and $\lim_{n \in \mathbb{Z}, |n| \rightarrow \infty} f(n)e^{4|n|} = 0$. Then $f \equiv 0$.

Proof. For each $x \in \mathbb{R}$, using Theorem 2.4 for $w \equiv 0$ and $r > |x|$, one has

$$\begin{aligned} |f(x)| &\leq \|f\|_{L^2(\mathbb{R})} e^{-r/2} + 8re^{3.96r} \max_{n \in A_r} |f(n)| \\ &\leq \|f\|_{L^2(\mathbb{R})} e^{-r/2} + 8re^{-0.04r} \max_{n \in A_r} |f(n)e^{4|n|}| \rightarrow 0 \text{ as } r \rightarrow \infty. \end{aligned}$$

Thus $f(x) = 0$ for all $x \in \mathbb{R}$ and hence $f \equiv 0$. ■

3. Applications

In this section, we return to consider heat problems (1) and (4). Our aim is to prove the uniqueness of the solution of each problem as well as to show a scheme to compute the solution numerically.

We first consider the system (1).

Theorem 3.1. (Uniqueness) *For each $f \in L^1(Q)$, the system (1) has at most one solution*

$$u \in C^1([0, T]; L^1(0, 1)) \cap L^2(0, T; H^2(0, 1)).$$

Proof. Assume that u_1 and u_2 are two solutions of (1) corresponding to the same data f . Then $u = u_1 - u_2$ satisfies (1) corresponding to $f = 0$. Then from (2) one has

$$\begin{aligned} |F(u(\cdot, T))(n)| e^{4|n|} &= |F(u(\cdot, 0))(n)| e^{-n^2 T + 4|n|} \\ &\leq \|u(\cdot, 0)\|_{L^1(0, 1)} e^{-n^2 T + 4|n|} \rightarrow 0 \text{ as } |n| \rightarrow \infty. \end{aligned}$$

It follows from Corollary 2.5 that $u(\cdot, T) \equiv 0$. Of course, we can replace T by any $t \in (0, T]$ to get $u(\cdot, t) \equiv 0$. Thus $u_1 \equiv u_2$. ■

Note that **the existence of a solution is not considered here. Instead, we assume that there is an exact solution $u \in C^1([0, T]; L^1(0, 1)) \cap L^2(0, T; H^2(0, 1))$ of the system (1) corresponding to the exact data $f_0 \in L^1(Q)$. Our purpose in the next theorem is to construct a numerical solution approximating $u(\cdot, T)$ from a given data $f_\varepsilon \in L^1(Q)$, which satisfies $\|f_\varepsilon - f_0\|_{L^1(Q)} \leq \varepsilon$. For $x \in \mathbb{R}$, we shall denote by $\lceil x \rceil$ the integer in the interval $(x, x + 1]$.**

Theorem 3.2. (Numerical solution) *Assume that $u_0(\cdot, T) \in H^2(0, 1)$. Define*

$$\begin{aligned} H_\varepsilon(y) &= \int_0^T \int_0^1 f_\varepsilon(x, t) e^{y^2(t-T)} \cos(yx) dx dt, \quad y \in \mathbb{R}, \\ r_\varepsilon &= \left\lceil \frac{2}{9} \ln \left(\frac{1}{\varepsilon} \right) \right\rceil, \quad M_\varepsilon = \sqrt{\frac{1}{\varepsilon}}, \\ A_\varepsilon &= \{\pm(r_\varepsilon + j) | j = 1, 2, \dots, 4r_\varepsilon\}, \end{aligned}$$

$$F_\varepsilon(n) = \begin{cases} L[A_\varepsilon; H_\varepsilon], & \text{if } 0 \leq n\pi < r_\varepsilon, \\ H_\varepsilon(n\pi), & \text{if } r_\varepsilon \leq n\pi \leq M_\varepsilon, \end{cases}$$

$$v_\varepsilon(x) = F_\varepsilon(0) + 2 \sum_{1 \leq n \leq M_\varepsilon} F_\varepsilon(n) \cos(n\pi x).$$

Then we have the error estimate

$$\|v_\varepsilon - u_0(\cdot, T)\|_{H^1(0,1)} \leq C_0 \left(\ln \left(\frac{1}{\varepsilon} \right) \right)^{3/2} \varepsilon^{1/9},$$

where C_0 stands for a constant depending only on u_0 .

Proof. Denote by v_0 the exact solution $u_0(\cdot, T)$ for short. In the following, we use C_0 as a universal constant which depends only on u_0 and T .

Step 1. Compare $F(v_0)(n\pi)$ and $F(v_\varepsilon)(n\pi)$ for $0 \leq n\pi \leq M_\varepsilon$.

Let $y \in \mathbb{R}$, $|y| \geq r_\varepsilon$. It follows from (2) that

$$|F(v_0)(y) - H_0(y)| = e^{-y^2 T} |F(u_0(\cdot, 0))(y)| \leq C_0 \varepsilon$$

due to $e^{-y^2 T} \leq e^{6T^2 - \frac{9}{2}|y|} \leq e^{6T^2 - \frac{9}{2}r_\varepsilon} \leq e^{6T^2} \varepsilon$. Moreover $|H_0(y) - H_\varepsilon(y)| \leq \varepsilon$. Thus

$$|F(v_0)(y) - H_\varepsilon(y)| \leq |F(v_0)(y) - H_0(y)| + |H_0(y) - H_\varepsilon(y)| \leq C_0 \varepsilon.$$

In particular, one has

$$|F(v_0)(n\pi) - F(v_\varepsilon)(n\pi)| \leq C_0 \varepsilon \quad \text{if } r_\varepsilon \leq n\pi \leq M_\varepsilon, \quad (16)$$

and

$$\max_{n \in A_\varepsilon} |F(v_0)(n) - H_\varepsilon(n)| \leq C_0 \varepsilon.$$

We deduce from the latter inequality and Theorem 2.4 that if $0 \leq y < r_\varepsilon$ then

$$|F(v_0)(y) - L[A_\varepsilon; H_\varepsilon](y)| \leq \|F(v_0)\|_{L^2(0,1)} e^{-r_\varepsilon/2} + 8r_\varepsilon e^{3.96r_\varepsilon} \cdot C_0 \varepsilon.$$

Choosing $y = n\pi$ and using $\varepsilon^{-2/9} \leq e^{r_\varepsilon} \leq e\varepsilon^{-2/9}$, we get

$$|F(v_0)(n\pi) - F(v_\varepsilon)(n\pi)| \leq C_0 \varepsilon^{1/9} \quad \text{if } 0 \leq n\pi < r_\varepsilon. \quad (17)$$

Step 2. Applying the Parseval-Plancherel equality, one has

$$\begin{aligned} & \|v_0 - v_\varepsilon\|_{H^1(0,1)}^2 \\ &= |F(v_0)(0) - F(v_\varepsilon)(0)|^2 + 2 \sum_{n=1}^{\infty} (1 + n^2 \pi^2) |F(v_0)(n\pi) - F(v_\varepsilon)(n\pi)|^2 \end{aligned}$$

$$\leq 4\pi^2 \sum_{n=0}^{\infty} n^2 |F(v_0)(n\pi) - F(v_\varepsilon)(n\pi)|^2. \quad (18)$$

It follows from (17) that

$$\begin{aligned} & \sum_{0 \leq n\pi < r_\varepsilon} n^2 |F(v_0)(n\pi) - F(v_\varepsilon)(n\pi)|^2 \\ & \leq r_\varepsilon \max_{0 \leq n\pi < r_\varepsilon} \left\{ n^2 |F(v_0)(n\pi) - F(v_\varepsilon)(n\pi)|^2 \right\} \\ & \leq r_\varepsilon \cdot r_\varepsilon^2 \left(C_0 \varepsilon^{\frac{1}{9}} \right)^2 = C_0^2 r_\varepsilon^3 \varepsilon^{\frac{2}{9}}. \end{aligned} \quad (19)$$

Similarly, using (16) we have

$$\begin{aligned} & \sum_{r_\varepsilon \leq n\pi \leq M_\varepsilon} n^2 |F(v_0)(n\pi) - F(v_\varepsilon)(n\pi)|^2 \\ & \leq M_\varepsilon \max_{r_\varepsilon \leq n\pi \leq M_\varepsilon} \left\{ n^2 |F(v_0)(n\pi) - F(v_\varepsilon)(n\pi)|^2 \right\} \\ & \leq M_\varepsilon \cdot M_\varepsilon^2 (C_0 \varepsilon)^2 = C_0^2 \sqrt{\varepsilon}. \end{aligned} \quad (20)$$

Since $v_0 \in H^2(0, 1)$ and $v_0'(0) = v_0'(1) = 0$, we have

$$F(v_0)(n\pi) = -\frac{1}{n\pi} \int_0^1 v_0'(x) \sin(n\pi x) dx = -\frac{1}{n^2 \pi^2} \int_0^1 v_0''(x) \cos(n\pi x) dx.$$

The Parseval-Plancherel equality implies

$$\|v_0''\|_{L^2}^2 = \sum_{n=1}^{\infty} n^4 \pi^4 |F(v_0)(n\pi)|^2.$$

Therefore

$$\begin{aligned} \sum_{n\pi > M_\varepsilon} n^2 |F(v_0)(n\pi) - F(v_\varepsilon)(n\pi)|^2 &= \sum_{n\pi > M_\varepsilon} n^2 |F(v_0)(n\pi)|^2 \\ &\leq \frac{1}{M_\varepsilon^2} \sum_{n\pi > M_\varepsilon} n^4 |F(v_0)(n\pi)|^2 \\ &\leq \frac{1}{M_\varepsilon^2 \pi^4} \|v_0''\|_{L^2(0,1)}^2 \\ &\leq C_0 \varepsilon. \end{aligned} \quad (21)$$

From (18), (19), (20), (21) we conclude that

$$\|v_0 - v_\varepsilon\|_{H^1(0,1)}^2 \leq C_0 r_\varepsilon^3 \varepsilon^{\frac{2}{9}} \leq C_0^2 (\ln(\varepsilon^{-1}))^3 \varepsilon^{\frac{2}{9}}.$$

The proof is complete. ■

Remark 3.3. If the condition $v_0 := u(\cdot, T) \in H^2(0, 1)$ is reduced to $v_0 \in H^1(0, 1)$ only, we have

$$\|v_0 - v_\varepsilon\|_{L^2} \leq C_0 (\ln(\varepsilon^{-1}))^{\frac{1}{2}} \varepsilon^{\frac{1}{9}}.$$

The proof is even simpler with some small changes in Step 2.

Now we return to consider the inverse source problem (4). Recall that the approximation (6) only works if $e^{-\alpha^2 T} / |D(\varphi)(\alpha)| \rightarrow 0$ "fast enough" as $|\alpha| \rightarrow +\infty$, where

$$D(\varphi)(\alpha) = \int_0^T e^{\alpha^2(t-T)} \varphi(t) dt.$$

To keep $|D(\varphi)(\alpha)| \rightarrow 0$ "slowly" as $|\alpha| \rightarrow +\infty$, we shall need a slight condition (H) on φ .

Condition (H) $\varphi \in L^1(0, T)$ and either $\liminf_{t \rightarrow T^-} \varphi(t) > 0$ or $\limsup_{t \rightarrow T^-} \varphi(t) < 0$ holds.

Remark 3.4. The class of the functions satisfying (H) is very large. For example, this condition holds if φ is continuous at $t = T$ and $\varphi(T) \neq 0$.

Lemma 3.5. *If φ satisfies (H) then*

$$\liminf_{|\alpha| \rightarrow \infty} \alpha^2 |D(\varphi)(\alpha)| > 0.$$

Proof. Assume that $\liminf_{t \rightarrow T^-} \varphi(t) > 0$. Then there exist $\lambda \in (0, T)$ and $C_0 > 0$ such that $\varphi(t) > C_0$ for all $t \in (\lambda, T)$. Therefore

$$\begin{aligned} |D(\varphi)(\alpha)| &\geq \int_\lambda^T e^{\alpha^2(t-T)} \varphi(t) dt - \left| \int_0^\lambda e^{\alpha^2(t-T)} \varphi(t) dt \right| \\ &\geq \int_\lambda^T e^{\alpha^2(t-T)} C_0 dt - \int_0^\lambda e^{\alpha^2(\lambda-T)} |\varphi(t)| dt \\ &\geq C_0 \frac{1 - e^{\alpha^2(\lambda-T)}}{\alpha^2} - e^{\alpha^2(\lambda-T)} \|\varphi\|_{L^1(0, T)}. \end{aligned}$$

Multiplying both sides of the latter inequality and then letting $|\alpha| \rightarrow \infty$, we obtain the desired result. \blacksquare

Theorem 3.6. (Uniqueness) *Assume that φ satisfies (H) and $g \in L^2(0, 1)$. Then the system (4) has at most one solution*

$$(u, f) \in (C^1([0, T]; L^1(0, 1)) \cap L^2(0, T; H^2(0, 1)), L^2(0, 1)).$$

Proof. Let (u_1, f_1) and (u_2, f_2) be two solutions. Put $u = u_1 - u_2$ and $f = f_1 - f_2$. Then (u, f) is a solution of (4) corresponding to φ and $g \equiv 0$. Therefore, (5) reduces to

$$e^{-\alpha^2 T} F(u(\cdot, 0))(n) = D(\varphi)(\alpha) \cdot F(f)(\alpha), \quad \alpha \in \mathbb{R}. \quad (22)$$

In particular, choosing $\alpha = n \in \mathbb{Z}$, one has

$$\left| e^{4|n|} F(f)(n) \right| \cdot |n^2 D(\varphi)(n)| = n^2 e^{-n^2 T + 4|n|} \|F(u(\cdot, 0))\|_{L^\infty} \rightarrow 0 \quad \text{as } |n| \rightarrow \infty.$$

On the other hand, $\liminf_{|n| \rightarrow \infty} |n^2 D(\varphi)(n)| > 0$ due to Lemma 3.5. Thus

$$e^{4|n|} |F(f)(n)| = 0 \rightarrow 0 \quad \text{as } |n| \rightarrow \infty.$$

It follows from Corollary 2.5 that $f \equiv 0$. Therefore, Equation (22) reduces to $F(u(\cdot, 0))(n) = 0$ for all $n \in \mathbb{Z}$, and hence $u(\cdot, 0) \equiv 0$. As known, the homogeneous heat equation with the trivial initial condition $u(\cdot, 0) = 0$ has only the trivial solution $u \equiv 0$. Thus $(u_1, f_1) = (u_2, f_2)$. \blacksquare

We mention that the problem (4) is severely ill-posed, and hence the regularization is necessary. Assume that

$$(u_0, f_0) \in (C^1([0, T]; L^1(0, 1)) \cap L^2(0, T; H^2(0, 1)), L^2(0, 1))$$

is the exact solution of (4) corresponding to the exact data (φ_0, g_0) , where φ_0 satisfies (H) and $g_0 \in L^2(0, 1)$. Let given data $(\varphi_\varepsilon, g_\varepsilon) \in (L^1(0, T), L^2(0, 1))$ satisfy

$$\|\varphi_\varepsilon - \varphi_0\|_{L^1(0, T)} \leq \varepsilon \quad \text{and} \quad \|g_\varepsilon - g_0\|_{L^2(0, 1)} \leq \varepsilon.$$

Our aim is to construct, from $(\varphi_\varepsilon, g_\varepsilon)$, a regularized solution f_ε approximating f_0 . We shall use a scheme similar to the one in Theorem 3.2.

Theorem 3.7. (Regularization) *Assume that $f_0 \in H^1(0, 1)$. Define*

$$\begin{aligned} H_\varepsilon(y) &= \begin{cases} \frac{F(g_\varepsilon)(y)}{D(\varphi_\varepsilon)(y)}, & \text{if } D(\varphi_\varepsilon)(y) \neq 0, \\ 0, & \text{if } D(\varphi_\varepsilon)(y) = 0, \end{cases} \\ r_\varepsilon &= \left\lceil \frac{2}{9} \ln \left(\frac{1}{\varepsilon} \right) \right\rceil, \quad M_\varepsilon = \pi^2 \sqrt[6]{\frac{1}{\varepsilon}}, \\ A_\varepsilon &= \{\pm(r_\varepsilon + j) \mid j = 1, 2, \dots, 4r_\varepsilon\}, \\ F_\varepsilon(n) &= \begin{cases} L[A_\varepsilon; H_\varepsilon], & \text{if } 0 \leq n\pi < r_\varepsilon, \\ H_\varepsilon(n\pi), & \text{if } r_\varepsilon \leq n\pi \leq M_\varepsilon, \end{cases} \\ f_\varepsilon(x) &= F_\varepsilon(0) + 2 \sum_{1 \leq n \leq M_\varepsilon} F_\varepsilon(n) \cos(n\pi x). \end{aligned}$$

Then we have the error estimate

$$\|f_\varepsilon - f_0\|_{L^2(0,1)} \leq C_0 \sqrt{\ln\left(\frac{1}{\varepsilon}\right)} \varepsilon^{1/9},$$

where C_0 stands for a constant depending only on the exact data.

Proof. Since the main idea of the proof is similar to the one of Theorem 3.2 we shall just give the sketch. It is of course sufficient to consider $\varepsilon > 0$ small enough.

Step 1. Compare $F(v_0)(n\pi)$ and $F(v_\varepsilon)(n\pi)$ for $0 \leq n\pi \leq M_\varepsilon$.

We first consider $y \in \mathbb{R}$ such that $r_\varepsilon \leq |y| \leq M_\varepsilon$. Let $\varepsilon > 0$ be small enough, due to Lemma 3.5 we have

$$|D(\varphi_0)(y)| \geq \frac{C_1}{y^2} \quad (23)$$

for some constant $C_1 > 0$ depending only on φ_0 , and

$$\begin{aligned} |D(\varphi_\varepsilon)(y)| &\geq |D(\varphi_0)(y)| - |D(\varphi_0)(y) - D(\varphi_\varepsilon)(y)| \\ &\geq \frac{C_1}{y^2} - \varepsilon \geq \frac{C_1}{2y^2}. \end{aligned} \quad (24)$$

It follows from (5) and (23) that

$$\begin{aligned} |F(f_0)(y) - H_0(y)| &= e^{-y^2 T} \left| \frac{F(u(\cdot, 0)(y))}{D(\varphi_0)(y)} \right| \\ &\leq \frac{y^2 e^{-y^2 T}}{C_1} \|F(u(\cdot, 0))\|_{L^\infty} \leq C_0 \varepsilon \end{aligned} \quad (25)$$

due to $e^{-|y|} \leq e^{-|r_\varepsilon|} \leq \varepsilon^{\frac{2}{5}}$. Moreover, using (23), (24), one has the following estimate from direct calculation

$$|H_0(y) - H_\varepsilon(y)| = \left| \frac{F(g_0)(y)}{D(\varphi_0)(y)} - \frac{F(g_\varepsilon)(y)}{D(\varphi_\varepsilon)(y)} \right| \leq C_0 y^4 \varepsilon. \quad (26)$$

Employing (25), (26) and the triangle inequality, one has

$$|F(f_0)(y) - H_0(y)| \leq C_0 y^4 \varepsilon \quad \text{if } r_\varepsilon \leq |y| \leq M_\varepsilon.$$

In particular, one has

$$|F(f_0)(n\pi) - F(f_\varepsilon)(n\pi)| \leq C_0 M_\varepsilon^4 \varepsilon = C_0 \pi^8 \varepsilon^{1/3} \quad \text{if } r_\varepsilon \leq n\pi \leq M_\varepsilon, \quad (27)$$

and

$$\max_{n \in A_\varepsilon} |F(f_0)(n) - H_\varepsilon(n)| \leq C_0 (5r_\varepsilon)^4 \varepsilon.$$

It follows from Theorem 2.4 and the latter inequality that if $0 \leq y < r_\varepsilon$ then

$$|F(f_0)(y) - L[A_\varepsilon; H_\varepsilon](y)| \leq \|F(f_0)\|_{L^2(0,1)} e^{-r_\varepsilon/2} + 8r_\varepsilon e^{3.96r_\varepsilon} C_0 (5r_\varepsilon)^4 \varepsilon.$$

Choosing $y = n\pi$ and using $\varepsilon^{-2/9} \leq e^{r\varepsilon} \leq e\varepsilon^{-2/9}$, we get

$$|F(f_0)(n\pi) - F(f_\varepsilon)(n\pi)| \leq C_0\varepsilon^{1/9} \quad \text{if } 0 \leq n\pi < r_\varepsilon. \quad (28)$$

Step 2. Applying the Parseval-Plancherel equality, one has

$$\begin{aligned} & \|f_0 - f_\varepsilon\|_{L^2(0,1)}^2 \\ &= |F(f_0)(0) - F(f_\varepsilon)(0)|^2 + 2 \sum_{n=1}^{\infty} |F(v_0)(n\pi) - F(v_\varepsilon)(n\pi)|^2. \end{aligned} \quad (29)$$

We have

$$\sum_{0 \leq n\pi < r_\varepsilon} |F(v_0)(n\pi) - F(v_\varepsilon)(n\pi)|^2 \leq C_0 r_\varepsilon \varepsilon^{2/9} \quad (30)$$

due to (27), and

$$\sum_{r_\varepsilon \leq n\pi \leq M_\varepsilon} |F(v_0)(n\pi) - F(v_\varepsilon)(n\pi)|^2 \leq C_0 \sqrt{\varepsilon} \quad (31)$$

due to (28). Finally

$$\begin{aligned} \sum_{n\pi > M_\varepsilon} |F(f_0)(n\pi) - F(f_\varepsilon)(n\pi)|^2 &= \sum_{n\pi > M_\varepsilon} |F(f_0)(n\pi)|^2 \\ &\leq \frac{1}{M_\varepsilon^2} \sum_{n\pi > M_\varepsilon} n^2 \pi^2 |F(f_0)(n\pi)|^2 \\ &\leq \|f'\|_{L^2}^2 \varepsilon^{1/3}. \end{aligned} \quad (32)$$

From (29), (30), (31), (32) we conclude that

$$\|f_0 - f_\varepsilon\|_{L^2}^2 \leq C_0^2 r_\varepsilon \varepsilon^{\frac{2}{9}} \leq C_0^2 \ln(\varepsilon^{-1}) \varepsilon^{\frac{2}{9}}.$$

The proof is complete. ■

Remark 3.8. The error estimate in Theorem 3.7, which is of order $\sqrt{\ln(\varepsilon^{-1})} \varepsilon^{1/9}$, is a significant improvement in comparison with the error estimate in the second regularization theorem of [17], which is of order $1/\ln(\varepsilon^{-1})$. Moreover, in [17] the authors had to require in addition the initial condition $u(., 0)$.

4. Numerical Experiment

To see the effect of our method, let us consider an example for the inverse source problem (4). Choose $T = 1$ and the exact data

$$\varphi_0(t) = e^{t-1}, \quad g_0(x) = 2x^3 - 3x^2 - \cos(\pi x).$$

Then the exact solution of (4) is

$$\begin{aligned} u_0(x, t) &= e^{t-1} (2x^3 - 3x^2 - \cos(\pi x)), \\ f_0(x, t) &= 2x^3 - 3x^2 - 12x + 6 - (1 + \pi^2) \cos(\pi x). \end{aligned}$$

For each integer $n \geq 1$, consider the disturbed data

$$\varphi_n(t) = \varphi_0(t), \quad g_n(x) = g_0(x) + \sqrt{\frac{8}{3}} \cdot \frac{\sin^2(n\pi x)}{n}.$$

Then the disturbed solution corresponding to the disturbed data is

$$\begin{aligned} \tilde{u}_n(x, t) &= u_0(x, t) + e^{t-1} \sqrt{\frac{8}{3}} \cdot \frac{\sin^2(n\pi x)}{n}, \\ \tilde{f}_n(x) &= f_0(x) + \sqrt{\frac{8}{3}} \cdot \frac{(4n^2\pi^2 + 1) \sin^2(n\pi x) - 2n^2\pi^2}{n}. \end{aligned}$$

We see that

$$\|g_n - g_0\|_{L^2(0,1)} = \frac{1}{n} \rightarrow 0 \quad \text{as } n \rightarrow +\infty$$

but

$$\|\tilde{f}_n - f_0\|_{L^2(0,1)} = \frac{\sqrt{16\pi^4 n^4 + 8\pi^2 n^2 + 3}}{\sqrt{3}n} \rightarrow +\infty \quad \text{as } n \rightarrow +\infty.$$

Thus if n is large then a small error of the data causes a large error of the solutions. Therefore, the problem is ill-posed and a regularization is necessary.

Now we use our scheme in Theorem 3.7 with respect to $\varepsilon = n^{-1}$ to compute the regularized solution. Corresponding to $\varepsilon_k = 10^{-k}$, we obtain the following regularized solutions

$$\begin{aligned} f_{\varepsilon_1}(x) &= -0.448859 - 5.513525 \cos(\pi x) + 0.546463 \cos(3\pi x), \\ f_{\varepsilon_2}(x) &= -0.5007813 - 5.513525 \cos(\pi x) + 0.546463 \cos(3\pi x) \\ &\quad + 0.195325 \cos(5\pi x), \\ f_{\varepsilon_3}(x) &= -0.512356 - 5.513525 \cos(\pi x) + 0.546463 \cos(3\pi x) \\ &\quad + 0.195325 \cos(5\pi x) + 0.099459 \cos(7\pi x) + 0.060117 \cos(9\pi x). \end{aligned}$$

The errors between regularized solutions and the exact solution are given by Table 1.

Fig. 4. gives a visual comparison between the regularized solution corresponding to $\varepsilon = \varepsilon_1 := 10^{-1}$ and the exact solution.

Note that the disturbed solution corresponding to $\varepsilon = \varepsilon_1 := 10^{-1}$ causes a large error $\|\tilde{f}_{\varepsilon_1} - f_0\|_{L^2} \approx 227.9865$.

$\varepsilon = n^{-1}$	$\ f_0 - f_\varepsilon\ _{L^2}$	$\frac{\ f_0 - f_\varepsilon\ _{L^2}}{\ f_0\ _{L^2}}$
$\varepsilon_1 = 10^{-1}$	0.1742974617	0.04409277801
$\varepsilon_2 = 10^{-2}$	0.09321430861	0.02358082429
$\varepsilon_3 = 10^{-3}$	0.04569371374	0.01155933515
$\varepsilon_4 = 10^{-4}$	0.02675238031	0.006767664629
$\varepsilon_5 = 10^{-5}$	0.01374434647	0.003476966399

Table 1 The error between regularized solutions and the exact solution

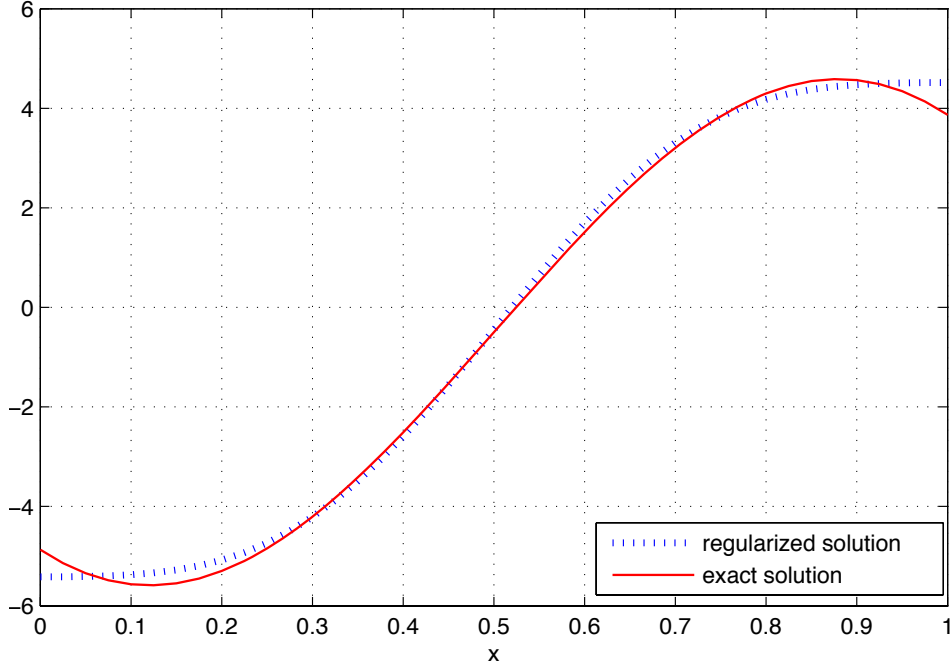


Fig. 1 The regularized solution f_{ε_1} and the exact solution

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