Stationary solutions to the Poisson-Nernst-Planck equations with steric effects

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L.-C. Hung dedicates this work to his beloved parents

Abstract

A method based on a differential algebraic equation (DAE) approach is employed to find stationary solutions of the Poisson-Nernst-Planck equations with steric effects (PNP-steric equations). Under appropriates boundary conditions, the equivalence of the PNP-steric equations and a corresponding system of DAEs is shown. Solving this system of DAEs leads to the existence of stationary solutions of PNP-steric equations. We show that for suitable range of the parameters, the steric effect can produces infinitely many discontinuous stationary solutions. Moreover under a stronger intra-species steric effect, we prove that a smooth solution converges to a constant stationary solution.

1 Introduction and statement of results

To study biological ion channels, the Poisson-Nernst-Planck system ([2], [4], [6], [8]) is commonly used as a model to describe the ionic flows in open ion channels. The Poisson-

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Nernst-Planck system is given by

$$\begin{cases} \dfrac{\partial c_n}{\partial t} = \dfrac{\partial}{\partial x} \left[D_n \left(\dfrac{\partial c_n}{\partial x} + z_n\, c_n\, \dfrac{\partial \phi}{\partial x} \right) \right], \\ \dfrac{\partial c_p}{\partial t} = \dfrac{\partial}{\partial x} \left[D_p \left(\dfrac{\partial c_p}{\partial x} + z_p\, c_p\, \dfrac{\partial \phi}{\partial x} \right) \right], \\ -\varepsilon\, \dfrac{\partial^2 \phi}{\partial x^2} = z_n\, c_n + z_p\, c_p - \rho, \end{cases} \tag{1.1}$$

for $x \in (-1,1)$ and $t > 0$. Here $c_n = c_n(x,t)$ is the density of anions and $c_p = c_p(x,t)$ is the density of cations; $\phi = \phi(x,t)$ is the electrostatic potential; D_n and D_p are the diffusion coefficients; $-z_n$ and z_p are positive integers; ρ is the permanent charge density; ε is a parameter. Next we introduce a new type of Poisson-Nernst-Planck system by considering the steric effect (or size effect) which occurs due to the fact that each atom within a molecule occupies a certain amount of space:

$$\begin{cases} \dfrac{\partial c_n}{\partial t} = \dfrac{\delta}{A(x)} \dfrac{\partial}{\partial x} \left[A(x) \left(\dfrac{\partial c_n}{\partial x} + z_n\, c_n\, \dfrac{\partial \phi}{\partial x} \right) \right] + \dfrac{1}{A(x)} \dfrac{\partial}{\partial x} \left[A(x) \left(g_{nn}\, c_n\, \dfrac{\partial c_n}{\partial x} + g_{np}\, c_n\, \dfrac{\partial c_p}{\partial x} \right) \right], \\ \dfrac{\partial c_p}{\partial t} = \dfrac{\delta}{A(x)} \dfrac{\partial}{\partial x} \left[A(x) \left(\dfrac{\partial c_p}{\partial x} + z_p\, c_p\, \dfrac{\partial \phi}{\partial x} \right) \right] + \dfrac{1}{A(x)} \dfrac{\partial}{\partial x} \left[A(x) \left(g_{pp}\, c_p\, \dfrac{\partial c_p}{\partial x} + g_{np}\, c_p\, \dfrac{\partial c_n}{\partial x} \right) \right], \\ -\dfrac{\varepsilon}{A(x)} \dfrac{\partial}{\partial x} \left(A(x) \dfrac{\partial \phi}{\partial x} \right) = z_n\, c_n + z_p\, c_p, \end{cases} \tag{1.2}$$

for $x \in (-1,1)$ and $t > 0$. Here $c_n = c_n(x,t)$ is the density of anions and $c_p = c_p(x,t)$ is the density of cations; $\phi = \phi(x,t)$ is the electrostatic potential; $A = A(x)$ is the cross-sectional area of the ion channel at position x; g_{nn}, g_{pp} and g_{np} are positive constants; $-z_n$ and z_p are positive integers; δ and ε are two parameters.

Assuming constant cross-sectional area $A(x)$, without loss of generality, we consider the following two-component *drift-diffusion system*

$$\begin{cases} u_t = \nabla \cdot (d_1\, \nabla u + \vartheta_1\, u\, \nabla \phi) + \nabla \cdot (g_{11}\, u\, \nabla u + g_{12}\, u\, \nabla v), \quad x \in \Omega, \quad t > 0, \\ v_t = \nabla \cdot (d_2\, \nabla v + \vartheta_2\, v\, \nabla \phi) + \nabla \cdot (g_{21}\, v\, \nabla u + g_{22}\, v\, \nabla v), \quad x \in \Omega, \quad t > 0, \\ -\Delta \phi = \gamma_1\, u + \gamma_2\, v, \quad x \in \Omega, \quad t > 0, \end{cases} \tag{1.3}$$

where $u = u(x,t)$ and $v = v(x,t)$ are densities of the two species u and v, which are assumed to be nonnegative functions; $\phi = \phi(x,t)$ is the electric potential; d_1 and d_2 are diffusion rates. Throughout this paper, ϑ_1, g_{11}, g_{12}, g_{21}, g_{22} and γ_1 are positive constants; ϑ_2 and γ_2 are negative constants, unless otherwise specified. $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary.

When the domain under consideration is extended to the entire space $\mathbb{R}^n$, (1.3) becomes

$$\begin{cases} u_t = \nabla \cdot (d_1\, \nabla u + \vartheta_1\, u\, \nabla \phi) + \nabla \cdot (g_{11}\, u\, \nabla u + g_{12}\, u\, \nabla v), \quad x \in \mathbb{R}^n, \quad t > 0, \\ v_t = \nabla \cdot (d_2\, \nabla v + \vartheta_2\, v\, \nabla \phi) + \nabla \cdot (g_{21}\, v\, \nabla u + g_{22}\, v\, \nabla v), \quad x \in \mathbb{R}^n, \quad t > 0, \\ -\Delta \phi = \gamma_1\, u + \gamma_2\, v, \quad x \in \mathbb{R}^n, \quad t > 0. \end{cases} \tag{1.4}$$

We note that, in the absence of $g_{11}\, u\, \nabla u$, $g_{12}\, u\, \nabla v$, $g_{21}\, u\, \nabla v$, and $g_{22}\, v\, \nabla v$, (1.3) reduces to the Keller-Segel type equations

$$\begin{cases} u_t = \nabla \cdot (d_1\, \nabla u + \vartheta_1\, u\, \nabla \phi), \quad x \in \mathbb{R}^n, \quad t > 0, \\ v_t = \nabla \cdot (d_2\, \nabla v + \vartheta_2\, v\, \nabla \phi), \quad x \in \mathbb{R}^n, \quad t > 0, \\ -\Delta \phi = \gamma_1\, u + \gamma_2\, v, \quad x \in \mathbb{R}^n, \quad t > 0. \end{cases} \tag{1.5}$$

Keller-Segel model is a classical model in chemotaxis introduced by Keller and Segel ([11]). For the last two decades, there has been considerable literature devoted to the Keller-Segel model. For the Keller-Segel model, we refer to the textbook [12, 13], review papers [9, 10] and references therein.

In this paper, we are concerned with stationary solutions to (1.3) and (1.4), i.e. with time-independent solutions to the following elliptic systems

$$\begin{cases} 0 = \nabla \cdot (d_1\, \nabla u + \vartheta_1\, u\, \nabla \phi) + \nabla \cdot (g_{11}\, u\, \nabla u + g_{12}\, u\, \nabla v), \quad x \in \Omega, \\ 0 = \nabla \cdot (d_2\, \nabla v + \vartheta_2\, v\, \nabla \phi) + \nabla \cdot (g_{21}\, v\, \nabla u + g_{22}\, v\, \nabla v), \quad x \in \Omega, \\ -\Delta \phi = \gamma_1\, u + \gamma_2\, v, \quad x \in \Omega. \end{cases} \tag{1.6}$$

Using the elementary fact that $\nabla(\log u) = \nabla u / u$, the first and second equations in (1.6) can be rewritten as

$$\begin{cases} 0 = \nabla \cdot \Big(u\, \nabla (d_1\, \log u + \vartheta_1\, \phi + g_{11}\, u + g_{12}\, v)\Big), \quad x \in \Omega, \\ 0 = \nabla \cdot \Big(v\, \nabla (d_2\, \log v + \vartheta_2\, \phi + g_{21}\, u + g_{22}\, v)\Big), \quad x \in \Omega. \end{cases} \tag{1.7}$$

It is readily seen that if we can find u, v and ϕ satisfying the *algebraic equations*

$$\begin{cases} d_1\, \log u + \vartheta_1\, \phi + g_{11}\, u + g_{12}\, v = c_1, \quad x \in \Omega, \\ d_2\, \log v + \vartheta_2\, \phi + g_{21}\, u + g_{22}\, v = c_2, \quad x \in \Omega, \end{cases} \tag{1.8}$$

where c_1 and c_2 are constants, then such u, v and ϕ automatically form a solution of (1.7). A natural question arises as to whether *any* solution of (1.7) also satisfies (1.8). It will be shown in Proposition 2.1 that the answer is indeed affirmative when certain appropriate boundary conditions are imposed on the solutions, i.e.

$$u\, F_1 \frac{\partial F_1}{\partial \nu} \le 0, \quad \text{on} \quad \partial\Omega, \tag{1.9}$$

and

$$v\, F_2 \frac{\partial F_2}{\partial \nu} \le 0, \quad \text{on} \quad \partial\Omega, \tag{1.10}$$

where $F_1 = d_1\, \log u + \vartheta_1\, \phi + g_{11}\, u + g_{12}\, v$ and $F_2 = d_2\, \log v + \vartheta_2\, \phi + g_{21}\, u + g_{22}\, v$. For instance, $\frac{\partial F_i}{\partial \nu} = 0 \;\; (i = 1, 2)$ on $\partial\Omega$, or the Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial \phi}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega. \tag{1.11}$$

As a consequence, our problem now turns to establishing the existence of solutions for the *differential algebraic equations* (DAEs):

$$\begin{cases} d_1 \log u + \vartheta_1 \phi + g_{11} u + g_{12} v = c_1, & x \in \Omega, \\ d_2 \log v + \vartheta_2 \phi + g_{21} u + g_{22} v = c_2, & x \in \Omega, \\ -\Delta \phi = \gamma_1 u + \gamma_2 v, \quad x \in \Omega. \end{cases} \tag{1.12}$$

To the best of the authors' knowledge, this is the first time such DAE approach has been employed to drift-diffusion systems such as (1.6). In Section 2, we investigate in Theorem 2.2 the existence and uniqueness of solutions $u = u(\phi)$ and $v = v(\phi)$ to (1.8) under the following hypothesis:

(H1) $g_{11} g_{22} - g_{12} g_{21} \geq 0$.

Note that (1.8) is a system of nonlinear algebraic equations for which *explicit solutions* expressed by the form $u = u(\phi)$ and $v = v(\phi)$ in general cannot exist. Due to **(H1)** however, the solution $u = u(\phi)$ and $v = v(\phi)$ of (1.8) *in implicit form* can be given *uniquely*. With the aid of Theorem 2.2, we are finally led to the following semilinear Poisson equation

$$-\Delta \phi = G(\phi), \quad x \in \Omega, \tag{1.13}$$

where $G(\phi) = \gamma_1 u(\phi) + \gamma_2 v(\phi)$. To establish existence of solutions of (1.13) under the zero Neumann boundary condition

$$\frac{\partial \phi}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.14}$$

more delicate properties of the nonlinearity $G = G(\phi)$ and the solution $u = u(\phi)$, $v = v(\phi)$ are explored in Proposition 2.3. These important properties turn out to be essential in proving the existence of solutions of (1.13) by the standard *direct method* in the calculus of variations.

Theorem 1.1 (**Existence of stationary solutions to PNP-steric equations under (H1)**)**.** *Assume that **(H1)** holds and c_1 and c_2 be fixed.* (1.6) *coupled with the Neumann boundary conditions* (1.11) *has a solution* $(u(x), v(x), \phi(x)) \in C^2(\Omega)$.

The intuition behind the differential algebraic equation approach we use in obtaining Theorem 1.1 is elementary. However, the result is remarkable in that only **(H1)** is needed to ensure the existence of solutions to the elliptic system (1.6) together with the Neumann boundary conditions (1.11). On the other hand, under the hypothesis:

(H2) $g_{11} g_{22} - g_{12} g_{21} < 0$,

(1.8) admits *triple solutions* (see Section 3). Moreover, in this case there may exist infinitely many solutions for (1.6).

Theorem 1.2 (Existence of stationary solutions to PNP-steric equations under (H2)). *Assume that **(H2)** holds and c_1 and c_2 be fixed.* (1.6) *coupled with the Neumann boundary conditions* (1.11) *has either a C^2 solution or infinitely many discontinuous solutions $(u(x), v(x), \phi(x))$.*

2 Two species equations

To begin with, we show that under the boundary conditions (1.9) and (1.10), every solution of (1.7) also solves (1.8), as mentioned in Introduction.

Proposition 2.1 (Equivalence of algebraic equations and differential equations). *The systems of PDEs*

$$\begin{cases} 0 = \nabla \cdot \Big(u \, \nabla (d_1 \, \log u + \vartheta_1 \, \phi + g_{11} \, u + g_{12} \, v) \Big), & x \in \Omega, \\ 0 = \nabla \cdot \Big(v \, \nabla (d_2 \, \log v + \vartheta_2 \, \phi + g_{21} \, u + g_{22} \, v) \Big), & x \in \Omega, \end{cases}$$

together with the boundary conditions

$$u \, F_1 \, \frac{\partial F_1}{\partial \nu} \leq 0, \quad on \quad \partial\Omega,$$

and

$$v \, F_2 \, \frac{\partial F_2}{\partial \nu} \leq 0, \quad on \quad \partial\Omega,$$

where $F_1 = d_1 \, \log u + \vartheta_1 \, \phi + g_{11} \, u + g_{12} \, v$ and $F_2 = d_2 \, \log v + \vartheta_2 \, \phi + g_{21} \, u + g_{22} \, v$, is equivalent to the system of algebraic equations

$$\begin{cases} d_1 \, \log u + \vartheta_1 \, \phi + g_{11} \, u + g_{12} \, v = c_1, & x \in \Omega, \\ d_2 \, \log v + \vartheta_2 \, \phi + g_{21} \, u + g_{22} \, v = c_2, & x \in \Omega. \end{cases}$$

Proof. Integration by parts leads to the desired result. Indeed,

$$\begin{aligned} \int_\Omega u \, F_1 \Delta F_1 \, dx &= - \int_\Omega \nabla (u \, F_1) \cdot \nabla F_1 \, dx + \int_{\partial\Omega} u \, F_1 \, \frac{\partial F_1}{\partial \nu} \, ds \\ &\leq - \int_\Omega u \, |\nabla F_1|^2 \, dx - \int_\Omega F_1 \, \nabla u \cdot \nabla F_1 \, dx. \end{aligned} \tag{2.1}$$

However, $\nabla \cdot (u \, \nabla F_1)$=0 gives $0 = F_1 \, \nabla \cdot (u \, \nabla F_1) = F_1 \, \nabla u \cdot \nabla F_1 + u \, F_1 \Delta F_1$. This shows that $\int_\Omega u \, |\nabla F_1|^2 \, dx \leq 0$, and thus F_1 should be a constant independent of x. In a similar manner, we can prove that F_2 is a constant independent of x from $\nabla \cdot (v \, \nabla F_2) = 0$. The proof of the converse is trivial. Hence the proof is finished. □

Since we have established Proposition 2.1, it is now necessary to investigate existence and uniqueness of solutions to (1.8). In particular, the solution of interest takes the form $(u, v) = (u(\phi), v(\phi))$.

Remark 2.1. When the zero Neumann boundary conditions $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial \phi}{\partial \nu} = 0$ on $\partial\Omega$ are considered, it is easy to see that these boundary conditions lead to $\frac{\partial F_1}{\partial \nu} = 0$ and $\frac{\partial F_2}{\partial \nu} = 0$ on $\partial\Omega$. However, note that the converse is not true. Since the first two equations of (1.3) are in divergence form, the total charges are conserved:

$$\bar{u} = \int_\Omega u(x,t)\,dx = \int_\Omega u(x,0)\,dx, \quad \bar{v} = \int_\Omega v(x,t)\,dx = \int_\Omega v(x,0)\,dx, \tag{2.2}$$

where $(u(x,t), v(x,t))$ is a solution of (1.3). It follows from the divergence theorem and the Poisson equation in (1.3) that in the case of the homogeneous Neumann boundary condition $\dfrac{\partial \phi}{\partial \nu} = 0$, we have

$$\gamma_1\,\bar{u} = -\gamma_2\,\bar{v}, \tag{2.3}$$

which is equivalent to the *electroneutrality condition*. Here γ_1 and γ_2 stand for the valence numbers of ions u and v, respectively.

Theorem 2.2 (Existence of solutions to (1.8)**).** *Assume **(H1)**. Then for any given $\phi = \phi_0 \in \mathbb{R}$, there exists a unique solution $(u, v) = (u_0, v_0)$ of*

$$\begin{cases} d_1\,\log u + \vartheta_1\,\phi + g_{11}\,u + g_{12}\,v = c_1, \\ d_2\,\log v + \vartheta_2\,\phi + g_{21}\,u + g_{22}\,v = c_2. \end{cases} \tag{2.4}$$

Moreover, (3.8) *has a unique solution (u, v, ϕ) which can be represented implicitly as $(u, v) = (u(\phi), v(\phi))$ and $u(\phi), v(\phi)$ are C^1 functions. Prove that*

Proof. For any given $\phi_0 \in \mathbb{R}$, we first show that the system of algebraic equations (3.8) has at least one solution $(u, v) = (u_0, v_0)$. In other words, the two curves $d_1\,\log u + g_{11}\,u + g_{12}\,v = c_1 - \vartheta_1\,\phi_0$ and $d_2\,\log v + g_{21}\,u + g_{22}\,v = c_2 - \vartheta_2\,\phi_0$ on the v-u plane has at least one intersection point. To see this, we differentiate $d_1\,\log u(v) + g_{11}\,u(v) + g_{12}\,v = c_1 - \vartheta_1\,\phi_0$ with respect to v to obtain

$$u'(v) = -\frac{g_{12}\,u(v)}{d_1 + g_{11}\,u(v)} < 0, \tag{2.5}$$

which is always strictly less than zero because of the term $\log u$ appearing in the first equation of (3.8) and u cannot take nonpositive value in $\log u$. Therefore, the profile $u = u(v)$ of $d_1\,\log u + g_{11}\,u + g_{12}\,v = c_1 - \vartheta_1\,\phi_0$ on the v-u plane has the following property:

- as $v \to -\infty$, $u \to \infty$;
- as $v \to \infty$, $u \to 0^+$;

- $u = u(v)$ is decreasing in $v \in \mathbb{R}$.

In a similar manner, we have for the second equation $d_2 \log v(u) + g_{21}\, u + g_{22}\, v(u) = c_2 - \vartheta_2\, \phi_0$ of (3.8),

$$v'(u) = -\frac{g_{21}\, v}{d_2 + g_{22}\, v} < 0. \tag{2.6}$$

The profile $v = v(u)$ of $d_2 \log v + g_{21}\, u + g_{22}\, v = c_2 - \vartheta_2\, \phi_0$ on the v-u plane enjoys the following property:

- as $u \to -\infty$, $v \to \infty$;
- as $u \to \infty$, $v \to 0^+$;
- $v = v(u)$ is decreasing in $u \in \mathbb{R}$.

As a consequence, it follows from the property of the profiles of the two curves $d_1 \log u + g_{11}\, u + g_{12}\, v = c_1 - \vartheta_1\, \phi_0$ and $d_2 \log v + g_{21}\, u + g_{22}\, v = c_2 - \vartheta_2\, \phi_0$, that the two curves in the first quadrant of the v-u plane intersect at least once. That is, given any $\phi = \phi_0 \in \mathbb{R}$, we can find at least one solution $(u, v) = (u_0, v_0)$ which satisfies (3.8). We eliminate the possibility of non-uniqueness of solutions (u, v) to (3.8) for a given $\phi \in \mathbb{R}$ by contradiction. Suppose that, contrary to our claim, there exist in the first quadrant of the v-u plane two distinct solutions (u_1, v_1) and (u_2, v_2) which satisfy (3.8) for given $\phi = \phi_0 \in \mathbb{R}$. Denote by $M_1(u, v)$ (respectively, $M_2(u, v)$) the slope of the curve $d_1 \log u + g_{11}\, u + g_{12}\, v = c_1 - \vartheta_1\, \phi_0$ (respectively, $d_2 \log v + g_{21}\, u + g_{22}\, v = c_2 - \vartheta_2\, \phi_0$) at (u, v). It is easy to observe that,

$$(M_1(u_1, v_1) - M_2(u_1, v_1))\; (M_1(u_2, v_2) - M_2(u_2, v_2)) < 0. \tag{2.7}$$

Without loss of generality, we may assume that $M_1(u_1, v_1) - M_2(u_1, v_1) < 0$ and $M_1(u_2, v_2) - M_2(u_2, v_2) > 0$. According to the Intermediate Value Theorem, there exists (u^*, v^*) for which $M_1(u^*, v^*) - M_2(u^*, v^*) = 0$, where u^* lies between u_1 and u_2 while v^* lies between v_1 and v_2. However, using (2.5) and (2.6), $M_1(u^*, v^*) = M_2(u^*, v^*)$ leads to

$$\frac{g_{12}\, u^*}{d_1 + g_{11}\, u^*} = \frac{d_2 + g_{22}\, v^*}{g_{21}\, v^*}. \tag{2.8}$$

It turns out that the last equation is equivalent to

$$\left(\frac{d_1}{u^*} + g_{11}\right)\left(\frac{d_2}{v^*} + g_{22}\right) = g_{12}\, g_{21}, \tag{2.9}$$

which contradicts **(H1)**. Consequently, for given $\phi \in \mathbb{R}$, uniqueness of solutions to (3.8) follows. By applying the local implicit function theorem at each point $(u(\phi), v(\phi), \phi)$ which satisfies (3.8), we obtain C^1 smoothness of the solution $(u, v) = (u(\phi), v(\phi))$. The proof is completed.

□

Certain important properties of solutions to (1.8) are investigated in the next proposition. These properties include non-simultaneousness of vanishing u and v, monotonicity of $u = u(\phi)$ and $v = v(\phi)$, and monotonicity and convexity of $u = u(v)$. An important consequence of these properties is (2.15), which turns out to play a crucial role in employing the direct method to establish existence of solutions for the semilinear Poisson equation (1.13).

Proposition 2.3 (**Properties of solutions to** (1.8))**.** *As in Theorem 2.2, assume* ***(H1)****. Then* (3.8) *is uniquely solvable by the implicit functions* $(u, v) = (u(\phi), v(\phi))$*. Concerning the properties of* $(u, v) = (u(\phi), v(\phi))$*, we have*

(i) ***(Non-simultaneous vanishing of*** $u = u(\phi)$ ***and*** $v = v(\phi)$***).*** *For fixed* $\phi = \phi_0$ *and* (u, v) *which solves* (3.8)*, for* $\delta > 0$ *one of the following holds:*

$$u \geq \delta, \quad v \geq \delta, \tag{2.10}$$

$$u \geq \delta, \quad v < \delta, \tag{2.11}$$

$$u < \delta, \quad v \geq \delta. \tag{2.12}$$

It is equivalent to say: there exists $\delta > 0$ *so that* $u < \delta$ *and* $v < \delta$ *cannot simultaneously be true;*

(ii) ***(Monotonicity of*** $u = u(\phi)$ ***and*** $v = v(\phi)$***)*** $u'(\phi)$ *and* $v'(\phi)$ *can be expressed in terms of* $u(\phi)$ *and* $v(\phi)$*, i.e.*

$$u'(\phi) = -\frac{g_{12}\,\theta_2 - \theta_1\left(\frac{d_2}{v(\phi)} + g_{22}\right)}{g_{12}\,g_{21} - \left(\frac{d_1}{u(\phi)} + g_{11}\right)\left(\frac{d_2}{v(\phi)} + g_{22}\right)}, \tag{2.13}$$

$$v'(\phi) = -\frac{g_{21}\,\theta_1 - \theta_2\left(\frac{d_1}{u(\phi)} + g_{11}\right)}{g_{12}\,g_{21} - \left(\frac{d_1}{u(\phi)} + g_{11}\right)\left(\frac{d_2}{v(\phi)} + g_{22}\right)}. \tag{2.14}$$

Also, $u'(\phi) < 0$ *and* $v'(\phi) > 0$ *for* $\phi \in \mathbb{R}$*, i.e.* $u = u(\phi)$ *is monotonically decreasing in* $\phi \in \mathbb{R}$*, while* $v = v(\phi)$ *is monotonically increasing in* $\phi \in \mathbb{R}$*. Furthermore,* $u'(\phi)$ *and* $v'(\phi)$ *are uniformly bounded for* $\phi \in \mathbb{R}$*. In addition, we have*

$$\gamma_1\,u'(\phi) + \gamma_2\,v'(\phi) \leq -k, \tag{2.15}$$

for some constant $k > 0$*;*

(iii) ***(Monotonicity and convexity of*** $u = u(v)$***)*** u *can be expressed by* $u = u(v)$*. Moreover,*

$$u'(v) = \frac{u(v)\,(d_2\,\theta_1 + g_{22}\,\theta_1\,v - g_{12}\,\theta_2\,v)}{v\,(d_1\,\theta_2 + g_{11}\,\theta_2\,u(v) - g_{21}\,\theta_1\,u(v))} < 0, \tag{2.16}$$

which implies competition between u *and* v*, and*

$$u''(v) = \frac{d_1\,\theta_2\,v^2\,u'(v)^2 - d_2\,\theta_1\,u^2(v)}{v^2\,u(v)\,(d_1\,\theta_2 + g_{11}\,\theta_2\,u(v) - g_{21}\,\theta_1\,u(v))} > 0. \tag{2.17}$$

Proof. We prove (i) by contradiction. Assume to the contrary that there exists $\delta > 0$ such that $u < \delta$ and $v < \delta$ hold simultaneously. As δ is sufficiently small, u, v are even smaller and $\log u, \log v < 0$. On the other hand, $\theta_1 \phi$ and $\theta_2 \phi$ have opposite signs since $\theta_1 > 0$ and $\theta_2 < 0$. Therefore, in (3.8) either $d_1 \log u$ cannot be balanced with $\theta_1 \phi$ or $d_2 \log v$ cannot be balanced with $\theta_2 \phi$. Either case leads to a contradiction.

To prove (ii), we first differentiate the two equations in (3.8) one by one with respect to ϕ, and obtain two equations in which the unknowns can be viewed as $u'(\phi)$ and $v'(\phi)$. Solving them gives $u'(\phi)$ and $v'(\phi)$ as stated in (ii). Due to **(H1)**, it is easy to see that $u'(\phi) < 0$ and $v'(\phi) > 0$ for $\phi \in \mathbb{R}$.

Using (i), when $u, v \geq \delta$, clearly it follows from (2.13) and (2.14) that

$$u'(\phi) \leq -\frac{g_{12}\,\theta_2 - \theta_1\left(\frac{d_2}{\infty} + g_{22}\right)}{g_{12}\,g_{21} - \left(\frac{d_1}{\delta} + g_{11}\right)\left(\frac{d_2}{\delta} + g_{22}\right)} = \frac{g_{12}\,\theta_2 - g_{22}\,\theta_1}{\left(\frac{d_1}{\delta} + g_{11}\right)\left(\frac{d_2}{\delta} + g_{22}\right) - g_{12}\,g_{21}} \tag{2.18}$$

and

$$v'(\phi) \geq -\frac{g_{21}\,\theta_1 - \theta_2\left(\frac{d_1}{\delta} + g_{11}\right)}{g_{12}\,g_{21} - \left(\frac{d_1}{\infty} + g_{11}\right)\left(\frac{d_2}{\infty} + g_{22}\right)} = \frac{g_{21}\,\theta_1 - \theta_2\left(\frac{d_1}{\delta} + g_{11}\right)}{g_{11}\,g_{22} - g_{12}\,g_{21}} \tag{2.19}$$

for $\phi \in \mathbb{R}$. As for the other two cases $u \geq \delta, v < \delta$ and $u < \delta, v \geq \delta$, it suffices to consider one of them since they are symmetric. Suppose that $u \geq \delta, v < \delta$. For $u \geq \delta, v < \delta_0 \ll \delta$, (2.13) and (2.14) lead to

$$\lim_{v \to 0^+} u'(\phi) = -\frac{\theta_1}{\left(\frac{d_1}{u(\phi)} + g_{11}\right)} \leq -\frac{\theta_1}{\left(\frac{d_1}{\delta} + g_{11}\right)} \tag{2.20}$$

and

$$\lim_{v \to 0^+} v'(\phi) = 0. \tag{2.21}$$

On the other hand, when $u \geq \delta, \delta_0 \leq v < \delta$, estimates of $u'(\phi)$ and $v'(\phi)$ similar to (2.18) and (2.19) are given by

$$u'(\phi) \leq -\frac{g_{12}\,\theta_2 - \theta_1\left(\frac{d_2}{\infty} + g_{22}\right)}{g_{12}\,g_{21} - \left(\frac{d_1}{\delta} + g_{11}\right)\left(\frac{d_2}{\delta_0} + g_{22}\right)} = \frac{g_{12}\,\theta_2 - g_{22}\,\theta_1}{\left(\frac{d_1}{\delta} + g_{11}\right)\left(\frac{d_2}{\delta_0} + g_{22}\right) - g_{12}\,g_{21}} \tag{2.22}$$

and

$$v'(\phi) \geq -\frac{g_{21}\,\theta_1 - \theta_2\left(\frac{d_1}{\delta} + g_{11}\right)}{g_{12}\,g_{21} - \left(\frac{d_1}{\infty} + g_{11}\right)\left(\frac{d_2}{\delta} + g_{22}\right)} = \frac{g_{21}\,\theta_1 - \theta_2\left(\frac{d_1}{\delta} + g_{11}\right)}{g_{11}\,\left(\frac{d_2}{\delta} + g_{22}\right) - g_{12}\,g_{21}}. \tag{2.23}$$

As a consequence, we conclude that there exists a constant $k > 0$ such thtat $\gamma_1\, u'(\phi) + \gamma_2\, v'(\phi) \leq -k$.

Multiplying the first equation in (3.8) by θ_2 and the second equation in (3.8) by θ_1, we obtain two equations. Subtraction of the two resulting equations gives an equation in terms of u and v. Implicit differentiation then gives the desired result in (iii).

□

Theorem 2.2 and Proposition 2.3 lead us to consider the Neumann problem for the semilinear Poisson equation (1.13), i.e.

$$\begin{cases} -\Delta\phi = G(\phi) & \text{in} \quad \Omega, \\ \dfrac{\partial\phi}{\partial\nu} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.24}$$

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. First of all, it can be shown by means of Proposition 2.3 that $G(\phi)$ satisfies **(A1)** and **(A2)** of Theorem 3.3. Upon using Theorems 2.2 and 3.3, we establish the existence of (1.12). Applying Proposition 2.1 completes the proof of Theorem 1.1. □

3 Triple solutions; g_{ij}-induced triple solutions

For the case where the hypothesis **(H2)** is assumed we can also establish an existence theorem for (1.12). In particular, in this case (1.8) admits a *triple solution* in the sense that for some ϕ_0, we can find *three* pairs of solutions $(u_i(\phi_0), v_i(\phi_0))$ $(i = 1, 2, 3)$. See Figure 3.1 for an example. This is essentially different from the case where **(H1)** holds.

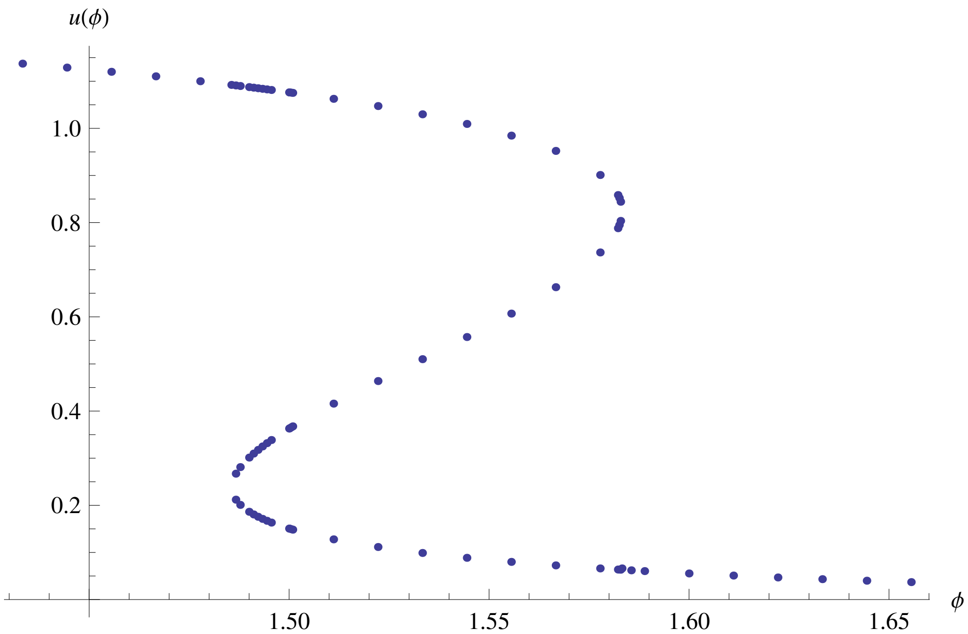

Figure 3.1: **A triple solution**.

As shown in Figure 3.1, there may exist a *triple solution* and thus uniqueness no longer holds when **(H1)** is violated, i.e. when $g_{11}\, g_{22} - g_{12}\, g_{21} < 0$. When non-uniqueness occurs, it is readily seen that there must be a $\phi^* \in \mathbb{R}$ such that $u'(\phi^*) = v'(\phi^*) = \infty$. Due to (2.13) and (2.14), this can occur only when $g_{12}\, g_{21} - \left(\frac{d_1}{u(\phi)} + g_{11}\right)\left(\frac{d_2}{v(\phi)} + g_{22}\right) = 0$. Indeed, it is possible that the numerator $g_{12}\,\theta_2 - \theta_1\left(\frac{d_2}{v(\phi)} + g_{22}\right) \to -\infty$ ($g_{21}\,\theta_1 - \theta_2\left(\frac{d_1}{u(\phi)} + g_{11}\right) \to -\infty$) in (2.13) ((2.14)) as $v(\phi) \to 0^+$ ($u(\phi) \to 0^+$). However, as we have mentioned in the proof of Proposition 2.3 that

$$\lim_{v\to 0^+} u'(\phi) = -\frac{\theta_1}{\left(\frac{d_1}{u(\phi)} + g_{11}\right)}, \quad \lim_{u\to 0^+} v'(\phi) = -\frac{\theta_2}{\left(\frac{d_2}{v(\phi)} + g_{22}\right)},$$

it is therefore necessary to find (u, v) at which $g_{12}\, g_{21} - \left(\frac{d_1}{u(\phi)} + g_{11}\right)\left(\frac{d_2}{v(\phi)} + g_{22}\right)$ vanishes by solving the equations:

$$\begin{cases} d_1\,\log u + \vartheta_1\,\phi + g_{11}\,u + g_{12}\,v = c_1, \\ d_2\,\log v + \vartheta_2\,\phi + g_{21}\,u + g_{22}\,v = c_2, \\ g_{12}\,g_{21} - \left(\frac{d_1}{u} + g_{11}\right)\left(\frac{d_2}{v} + g_{22}\right) = 0. \end{cases} \tag{3.1}$$

It turns out that (3.1) is reduced to a single equation $\sigma(u) = 0$ by eliminating ϕ and substituting $v = \frac{d_2\,(d_1 + g_{11}\,u)}{u\,(g_{12}\,g_{21} - g_{11}\,g_{22}) - d_1\,g_{22}}$. More precisely,

$$\begin{aligned} \sigma(u) =& \frac{1}{\theta_1}\left(c_1 - d_1\,\log u - g_{11}\,u - g_{12}\,\frac{d_2\,(d_1 + g_{11}\,u)}{u\,(g_{12}\,g_{21} - g_{11}\,g_{22}) - d_1\,g_{22}}\right) - \\ & \frac{1}{\theta_2}\left(c_2 - d_2\,\log\left(\frac{d_2\,(d_1 + g_{11}\,u)}{u\,(g_{12}\,g_{21} - g_{11}\,g_{22}) - d_1\,g_{22}}\right)\right. \\ & \left. \qquad - g_{21}\,u - g_{22}\,\frac{d_2\,(d_1 + g_{11}\,u)}{u\,(g_{12}\,g_{21} - g_{11}\,g_{22}) - d_1\,g_{22}}\right) \end{aligned}$$

Now the question remains to determine the profile of $\sigma(u) = 0$. To this end, we first observe that $\sigma(u)$ makes sense only when $u > u^*$, where $u^* := \frac{d_1\,g_{22}}{g_{12}\,g_{21} - g_{11}\,g_{22}} > 0$. Also, it is readily verified that

$$\lim_{u\to(u^*)^+} \sigma(u) = \lim_{u\to\infty} \sigma(u) = -\infty. \tag{3.2}$$

To find extreme points of $\sigma(u) = 0$, we calculate

$$\sigma'(u) = \frac{p(u)\,\left(u\,(g_{21}\,\theta_1 - g_{11}\,\theta_2) - d_1\,\theta_2\right)}{\theta_1\,\theta_2\,u\,\left(d_1 + g_{11}\,u\right)\,\left(u\,(g_{12}\,g_{21} - g_{11}\,g_{22}) - d_1\,g_{22}\right)^2}, \tag{3.3}$$

where $p(u) = k_3\, u^3 + k_2\, u^2 + k_1\, u + k_0$, and

$$
\begin{aligned}
k_3 &= g_{11}\, (g_{11}\, g_{22} - g_{12}\, g_{21})^2\,,\\
k_2 &= d_1\, (g_{12}\, g_{21} - 3\, g_{11}\, g_{22})\, (g_{12}\, g_{21} - g_{11}\, g_{22})\,,\\
k_1 &= d_1\, \left(d_2\, g_{21}\, g_{12}^2 + 2\, d_1\, g_{21}\, g_{22}\, g_{12} - 3\, d_1\, g_{11}\, g_{22}^2\right)\,,\\
k_0 &= d_1^3\, g_{22}^2.
\end{aligned}
\tag{3.4}
$$

We remark that the denominator of $\sigma'(u)$ is always away from 0 since $u > u^*$. Let the numerator of (3.3) be zero. Then the four roots are $0 > \frac{d_1\,\theta_2}{g_{21}\,\theta_1 - g_{11}\,\theta_2}$, u_1, u_2, and u_3, where u_1, u_2, and u_3 are the three roots of $p(u) = 0$. Indeed, u_1, u_2, and u_3 are three distinct real roots. To show this, we use Fan Shengjin's method. As in [7], let

$$
A = k_2^2 - 3\, k_1\, k_3, \quad B = k_1\, k_2 - 9\, k_0\, k_3, \quad C = k_1^2 - 3\, k_0\, k_2 \tag{3.5}
$$

and the *discriminant*

$$
\Delta_{dis} = B^2 - 4\, A\, C. \tag{3.6}
$$

Lemma 3.1 (Shengjin's discriminant([7])). *There are three possible cases using the discriminant Δ_{dis}:*

(i) If $\Delta_{dis} > 0$, then $p(u) = 0$ has one real root and two nonreal complex conjugate roots.

(ii) If $\Delta_{dis} = 0$, then $p(u) = 0$ has three real roots with one root which is at least of multiplicity 2.

(iii) If $\Delta_{dis} < 0$, then $p(u) = 0$ has three distinct real roots.

It can be shown that $\Delta_{dis} \geq 0$ cannot be true under the assumption **(H2)**. Using Lemma 3.1, we conclude that the cubic equation

$$
p(u) = k_3\, u^3 + k_2\, u^2 + k_1\, u + k_0 = 0 \tag{3.7}
$$

has three distinct real roots $u_3 < u_2 < u_1$. Due to $p(\pm\infty) = \pm\infty$ and $k_0, k_3 > 0$, it is easy to see that either $u_3 < u_2 < u_1 < 0$ or $u_3 < 0 < u_2 < u_1$. However, the case $u_3 < u_2 < u_1 < 0$ cannot occur since $\sigma(u)$ makes sense only for $u > u^* > 0$. For the other case $u_3 < 0 < u_2 < u_1$, u_3 cannot be a extreme point of $\sigma(u)$ because of $u_3 < 0$. Accordingly, there are *at most two* extreme points u_1 and u_2. We have by (3.2) the asymptotic behavior $\sigma(u) \to -\infty$ as $u \to u^*$ or $u \to \infty$, which leads to the fact that the number of extreme points of $\sigma(u) = 0$ can only be *odd*. As a consequence, one of u_1 and u_2 cannot be a extreme point of $\sigma(u) = 0$ and the other one is *the* extreme point of

$\sigma(u) = 0$. Suppose that u_2 is a extreme point of $\sigma(u) = 0$, then so is u_1, which yields a contradiction. Therefore, u_1 is *the* extreme point of $\sigma(u) = 0$. In fact, it can be shown that $u_2 \le u^*$ so that $\sigma(u_2)$ dose not make sense. Now a question remains, i.e., how to determine the sign of $\sigma(u_1)$? We see that the maximum of $\sigma(u)$ is attained at $u = u_1$ or $\max_{u>u^*} \sigma(u) = \sigma(u_1)$. Moreover, the criterion for determining the roots of the equation $\sigma(u) = 0$ is stated as:

- when $\sigma(u_1) > 0$, $\sigma(u) = 0$ has two distinct positive solutions;
- when $\sigma(u_1) < 0$, $\sigma(u) = 0$ has no solutions;
- when $\sigma(u_1) = 0$, $\sigma(u) = 0$ has a unique positive solution (i.e. $u = u_1$).

Noting that as $\sigma(u_1) = 0$ and $\sigma'(u_1) = 0$, we have $u'(\phi) = -v'(\phi) = u''(\phi) = -v''(\phi) = \infty$ evaluated at $u = u_1$. In other words $u(\phi)$ and $v(\phi)$ has a *reflection point* at $\phi = \check{\phi}$ for some $\check{\phi} \in \mathbb{R}$.

On the other hand, as $\sigma(u_1) < 0$, the denominator $g_{12}\, g_{21} - \left(\frac{d_1}{u(\phi)} + g_{11}\right)\left(\frac{d_2}{v(\phi)} + g_{22}\right)$ of (2.13) (also (2.14)) always keeps it sign since $\sigma(u) = 0$ has no solutions. Indeed, it is easy to see that $g_{12}\, g_{21} - \left(\frac{d_1}{u(\phi)} + g_{11}\right)\left(\frac{d_2}{v(\phi)} + g_{22}\right) < 0$, which leads to $u'(\phi) < 0$ and $v'(\phi) > 0$.

Theorem 3.2 (Trichotomy under (H2)). *Assume that* ***(H2)*** *holds and* $(u(\phi), v(\phi))$ *solves*

$$\begin{cases} d_1 \,\log u + \vartheta_1\, \phi + g_{11}\, u + g_{12}\, v = c_1, \\ d_2 \,\log v + \vartheta_2\, \phi + g_{21}\, u + g_{22}\, v = c_2. \end{cases} \tag{3.8}$$

Then

(i) ***(triple solutions)*** *when* $\sigma(u_1) > 0$, *there exist* $\underline{\phi}, \bar{\phi} \in \mathbb{R}$ *such that*

- $u(\phi)$ *(and* $v(\phi)$*) takes three distinct values for* $\phi \in (\underline{\phi}, \bar{\phi})$*;*
- $u(\phi)$ *(and* $v(\phi)$*) can be represented uniquely for* $\phi \in (-\infty, \underline{\phi}) \cup (\bar{\phi}, \infty)$*;*
- $u(\phi)$ *(and* $v(\phi)$*) takes two distinct values at* $\phi = \underline{\phi}, \bar{\phi}$*;*

(ii) ***(uniqueness of monotone solutions)*** *when* $\sigma(u_1) < 0$, $u(\phi)$ *and* $v(\phi)$ *can be represented uniquely for* $\phi \in \mathbb{R}$. *Moreover,* $u'(\phi) < 0$ *and* $v'(\phi) > 0$ *for* $\phi \in \mathbb{R}$*;*

(iii) ***(unique and monotone solutions with inflection points)*** *when* $\sigma(u_1) = 0$, $u(\phi)$ *and* $v(\phi)$ *can be represented uniquely for* $\phi \in \mathbb{R}$. *Furthermore, there exists* $\check{\phi} \in \mathbb{R}$ *such that*

- $u'(\phi) < 0$ *and* $v'(\phi) > 0$ *for* $\phi \in (-\infty, \check{\phi}) \cup (\check{\phi}, \infty)$*;*
- $u'(\phi) = -v'(\phi) = \infty$ *at* $\phi = \check{\phi}$.

We are able to explain c_1 c_2 in the following manner. According to Theorem 3.2, the sign of $\sigma(u_1)$ determines the properties, such as the number of solutions, of (3.8). Here u_1 is the largest positive root of $p(u) = 0$ and the discriminant as shown in (3.5) relies on d_i and g_{ij} $(i, j = 1, 2)$. Consequently, u_1 is determined once d_i and g_{ij} $(i, j = 1, 2)$ are given. Now c_i and θ_i $(i, j = 1, 2)$ can be suitably chosen such that anyone of the three cases as described in Theorem 3.2 can occur.

Theorem 3.3 (**Existence of solutions to Poission equations with discontinuous nonlinearity**). *Assume that $\Omega \subset \mathbb{R}^n$ is an open and bounded domain with smooth boundary $\partial\Omega$ and*

(A1) *$\rho'(u)$ is piecewise continuous and discontinuous at finite points for $u > 0$; $\left|\int_{\Omega^*} \rho'(s)\, ds\right| < \infty$ for all $\Omega^* \subseteq \mathbb{R}$;*

(A2) *there exist constants $K, L > 0$, such that $|\rho'(u)| \leq L$ for $|u| < K$ and $|\rho(u)| \geq \frac{1}{2} u^2$ for $|u| \geq K$.*

Then the following Neumann problem

$$\begin{cases} \Delta u = \rho'(u) & in \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & on \quad \partial\Omega, \end{cases} \tag{3.9}$$

has a unique weak solution $u \in H^1(\Omega)$.

Proof. For $u_0 \in \mathbb{R}$, let

$$\psi(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \int_\Omega \big(\rho(u) - \rho(u_0)\big) \, dx. \tag{3.10}$$

As seen previously, it is sufficient to consider the minimizing problem

$$\min_{u \in H^1(\Omega)} \psi(u). \tag{3.11}$$

Using **(A2)**, we have

$$\begin{aligned} \psi(u) &\geq \int_\Omega \big(\rho(u) - \rho(u_0)\big) \, dx \\ &\geq \int_{|u|<K} \rho(u) \, dx + \int_{|u| \geq K} \frac{1}{2} u^2 \, dx - \rho(u_0) \, |\Omega| \\ &\geq \int_{|u|<K} \rho(u) \, dx - \rho(u_0) \, |\Omega| \\ &= \int_{|u|<K} \left(\int_0^u \rho'(s) \, ds + \rho(0) \right) dx - \rho(u_0) \, |\Omega| \\ &\geq -\left(K\, L + |\rho(0)| + \rho(u_0)\right) |\Omega|. \end{aligned} \tag{3.12}$$

Therefore, there exists $M \in \mathbb{R}$ such that

$$M := \inf_{u \in H^1(\Omega)} \psi(u) > -\infty, \tag{3.13}$$

and we can find a sequence of functions $\{u_n(x)\}_{n=1}^{n=\infty} \in H^1(\Omega)$ with $\lim_{n\to\infty} \psi(u_n) = M$. Now we show that $\psi(u)$ is coercive, that is

$$\lim_{||u||\to\infty} \psi(u) = \infty, \tag{3.14}$$

where $||u|| = ||u||_{L^2(\Omega)} + ||\nabla u||_{L^2(\Omega)}$. Indeed, when $||\nabla u_n||_{L^2(\Omega)} \to \infty$ as $n \to \infty$, we clearly have (3.14). On the other hand, when $||\nabla u_n||_{L^2(\Omega)}$ is bounded and $||u_n||_{L^2(\Omega)} \to \infty$ as $n \to \infty$, it follows that (3.14) also holds. To see this,

$$\begin{aligned} \psi(u_n) &\geq \int_\Omega \big(\rho(u_n) - \rho(u_0)\big)\, dx \\ &= \int_{|u_n|<K} \rho(u_n)\, dx + \int_{|u_n|\geq K} \rho(u_n)\, dx - \rho(u_0)\,|\Omega| \\ &\geq -K\,L\,|\Omega| + \frac{1}{2}\int_{|u_n|\geq K} u_n^2\, dx - \rho(u_0)\,|\Omega|. \end{aligned} \tag{3.15}$$

Since $||u_n||_{L^2(\Omega)} \to \infty$ as $n \to \infty$ and $\int_{|u_n|<K} u_n^2\, dx \leq K^2\,|\Omega|$, we conclude that $\int_{|u_n|\geq K} u_n^2\, dx \to \infty$ and $\psi(u_n) \to \infty$ as $n \to \infty$. The coercivity condition (3.14) leads to the fact that there exists a constant $M_0 > 0$ such that $||u_n|| \leq M_0$ (otherwise, the unboundness of $||u_n||$ together with (3.13) contradicts (3.14)). It follows that the sequence $\{u_n(x)\}_{n=0}^{n=\infty}$ is bounded in $H^1(\Omega)$, and there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ with

$$u_{n_j} \rightharpoonup u^* \quad \text{in} \quad H^1(\Omega) \quad \text{as} \quad j \to \infty. \tag{3.16}$$

We select a subsequence $\{u_{n_{j_k}}\}$ of $\{u_{n_j}\}$ which converges in $L^2(\Omega)$, i.e.

$$u_{n_{j_k}} \to u^* \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad k \to \infty. \tag{3.17}$$

Since L^p convergence implies pointwise convergence of a subsequence almost everywhere, we have

$$u_{n_{j_{k_l}}} \to u^* \quad \text{a.e.} \quad \text{in} \quad \Omega \quad \text{as} \quad l \to \infty, \tag{3.18}$$

where $\{u_{n_{j_{k_l}}}\}$ is a subsequence of $\{u_{n_{j_k}}\}$. We denote this subsequence obtained by $\{u_k\}_{k=1}^{k=\infty}$. To find a minimizer of our minimizing problem, it suffices to establish weak lower semicontinuity, i.e.

$$\psi(u^*) \leq \liminf_{k\to\infty} \psi(u_k), \tag{3.19}$$

which gives

$$\begin{aligned} M &= \lim_{k\to\infty} \psi(u_k) \\ &\geq \liminf_{k\to\infty} \psi(u_k) \\ &= \liminf_{k\to\infty} \left(\frac{1}{2}\int_\Omega |\nabla u_k|^2\, dx + \int_\Omega \big(\rho(u_k) - \rho(u_0)\big)\, dx\right) \\ &\geq \frac{1}{2}\int_\Omega |\nabla u^*|^2\, dx + \int_\Omega \big(\rho(u^*) - \rho(u_0)\big)\, dx \\ &= \psi(u^*) \geq \inf_{u\in H^1(\Omega)} \psi(u) = M. \end{aligned} \tag{3.20}$$

This shows that u^* solves the minimizing problem (3.11). Now we prove (3.19). Indeed,

$$\rho(u) = \int_0^u \rho'(s)\, ds + \rho(0)$$

is continuous since $\rho'(s) \in L^1(\Omega^*)$ for all $\Omega^* \subseteq \mathbb{R}$. Since $u_k(x) \to u^*(x)$ a.e. in Ω as $k \to \infty$, we have $\rho(u_k) \to \rho(u^*)$ a.e.. Applying Fatou's lemma to obtain $\int_\Omega \rho(u^*)\, dx = \int_\Omega \liminf_{k\to\infty} \rho(u_k)\, dx \leq \liminf_{k\to\infty} \int_\Omega \rho(u_k)\, dx$ yields

$$\begin{aligned}
&\liminf_{k\to\infty} \psi(u_k) \\
&= \liminf_{k\to\infty} \left(\frac{1}{2} \int_\Omega |\nabla u_k|^2\, dx + \int_\Omega \big(\rho(u_k) - \rho(u_0)\big)\, dx \right) \\
&\geq \frac{1}{2} \int_\Omega |\nabla u^*|^2\, dx + \int_\Omega \big(\rho(u^*) - \rho(u_0)\big)\, dx = \psi(u^*),
\end{aligned} \tag{3.21}$$

where we have used the fact that $\int_\Omega |\nabla u_k|^2\, dx \to \int_\Omega |\nabla u^*|^2\, dx$ as $k \to \infty$. In fact, elliptic regularity of weak solutions ensure smoothness of u^*. This completes the proof of the theorem. □

4 Trend to equilibrium

In this section, the following initial conditions

$$u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0, \quad x \in \Omega, \tag{4.1}$$

are imposed to determine completely the evolution. Since the first two equations of (1.3) are in divergence form, total charges are conserved:

$$\bar{u} = \int_\Omega u(x,t)\, dx = \int_\Omega u_0(x)\, dx, \quad \bar{v} = \int_\Omega v(x,t)\, dx = \int_\Omega v_0(x)\, dx. \tag{4.2}$$

The means $\bar{w}_1$ and $\bar{w}_2$ of the charges u and v over Ω are defined by

$$\bar{w}_1 = \frac{\bar{u}}{|\Omega|}, \quad \bar{w}_2 = \frac{\bar{v}}{|\Omega|}, \tag{4.3}$$

respectively. The following two Lemmas are crucial in proving Theorem 4.3. We note that Csiszár-Kullback-Pinsker inequality (references refer to the ones cited in [3]) comes from information theory.

Lemma 4.1 (**Csiszár-Kullback-Pinsker inequality**). *For $u, v \in L^1(\Omega)$, $u, v \geq 0$,*

$$\int_\Omega \left(u \log\left(\frac{u}{v}\right) - u + v \right) dx \geq C_K(\Omega) \left(\int_\Omega |u - v|\, dx \right)^2, \tag{4.4}$$

where $C_K(\Omega)$ is a constant.

Lemma 4.2 (Logarithmic Sobolev inequality ([1])). *For $\sqrt{u} \in H^1(\Omega)$, $u \geq 0$,*

$$\int_\Omega u \log \frac{u}{\bar{w}}\, dx + C(n,\pi) \leq C_L(\Omega) \int_\Omega |\nabla \sqrt{u}|^2\, dx, \tag{4.5}$$

where $\bar{w} = \frac{\bar{u}}{|\Omega|}$, $\bar{u} = \int_\Omega u\, dx$, $C_L(\Omega)$ is the Logarithmic Sobolev constant, and $C(n,\pi)$ is a constant depending on n and π.

Consider the initial value problem for (1.3) with initial conditions $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$, $\phi(x,0) = \phi_0(x)$ and the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial \phi}{\partial \nu} = 0, \quad x \in \partial\Omega, \quad t > 0. \tag{4.6}$$

Under certain hypotheses on the initial conditions and the parameters appearing in (1.3), we show that the solution of the initial-boundary value problem (1.3) tends to $\bar{w}_1$ and $\bar{w}_1$ in the L^1 sense. More precisely, we have

Theorem 4.3 (Trend to equilibrium). *Assume*

($\mathcal{H}$1) $\frac{C_P^{-1}}{2}(2\, g_{11} - g_{12} - g_{21}) > \max\big(-\frac{1}{2}(\gamma_2\, \vartheta_1 + \gamma_1\, (2\, \vartheta_1 + \vartheta_2)), 0\big)$;

($\mathcal{H}$2) $\frac{C_P^{-1}}{2}(2\, g_{22} - g_{12} - g_{21}) > \max\big(-\frac{1}{2}(\gamma_1\, \vartheta_2 + \gamma_2\, (2\, \vartheta_2 + \vartheta_1)), 0\big)$;

($\mathcal{H}$3) $0 < H_0 < \infty$, *where* $H_0 = \displaystyle\int_\Omega \Big(u_0 \log \frac{u_0}{\bar{w}_1} + v_0 \log \frac{v_0}{\bar{w}_2}\Big)\, dx.$

*In **($\mathcal{H}$1)** and **($\mathcal{H}$2)**, C_P is the Poincaré constant. If a global-in-time solution $(u(x,t), v(x,t))$ of the time-varying problem* (1.3), (4.1), (4.6) *exists, then it tends to the corresponding constant steady-state solution $(\bar{w}_1, \bar{w}_2)$ in the L^1-norm as $t \to \infty$, i.e.*

$$\|u - \bar{w}_1\|_{L^1(\Omega)}, \|v - \bar{w}_2\|_{L^1(\Omega)} \longrightarrow 0, \quad as \quad t \longrightarrow \infty. \tag{4.7}$$

Furthermore, the time-varying solution $(u(x,t), v(x,t))$ converges in the L^1-norm exponentially fast to its mean with explicit rate $\mathcal{B}$:

$$\|u - \bar{w}_1\|^2_{L^1(\Omega)} + \|v - \bar{w}_2\|^2_{L^1(\Omega)} \leq C_K{}^{-1}\, H_0\, e^{-\mathcal{B}\, t}, \tag{4.8}$$

where $\mathcal{B}$ will be specified in the proof, and C_K is the constant in the Csiszár-Kullback-Pinsker inequality (Lemma 4.1).

Proof. As in [5], we define the *relative entropy functional* $H[u,v](t)$ by

$$H[u,v](t) = \int_\Omega \Big(u \log \frac{u}{\bar{w}_1} + v \log \frac{v}{\bar{w}_2}\Big)\, dx. \tag{4.9}$$

By virtue of the elementary fact $\log z + \frac{1}{z} - 1 \geq 0$ for $z > 0$, it can be shown that $H[u,v](t) \geq 0$ for all $t > 0$. Indeed,

$$\begin{aligned} 0 \leq & \int_\Omega u \Big(\log \frac{u}{\bar{w}_1} + \frac{\bar{w}_1}{u} - 1\Big) + v \Big(\log \frac{v}{\bar{w}_2} + \frac{\bar{w}_2}{v} - 1\Big)\, dx \\ = & H[u,v](t) + \int_\Omega \bar{w}_1\, dx - \int_\Omega u\, dx + \int_\Omega \bar{w}_2\, dx - \int_\Omega v\, dx \\ = & H[u,v](t). \end{aligned} \tag{4.10}$$

Note in particular that $\int_\Omega (u \log \frac{u}{\bar w_1})\, dx$, $\int_\Omega (v \log \frac{v}{\bar w_2})\, dx \ge 0$ for all $t > 0$. For simplicity of notation, let $\mathcal{I}_1 = d_1 \nabla u + \vartheta_1 u \nabla\phi + g_{11} u \nabla u + g_{12} u \nabla v$ and $\mathcal{I}_2 = d_2 \nabla v + \vartheta_2 v \nabla\phi + g_{21} v \nabla u + g_{22} v \nabla v$. We calculate the time derivative of $H[u, v]$ to obtain

$$\begin{aligned}
\partial_t H[u,v] &= \int_\Omega u_t \Big(\log \frac{u}{\bar w_1} + 1\Big) + v_t \Big(\log \frac{v}{\bar w_2} + 1\Big)\, dx \\
&= \int_\Omega \nabla\cdot\mathcal{I}_1 \Big(\log \frac{u}{\bar w_1} + 1\Big) + \nabla\cdot\mathcal{I}_1 \Big(\log \frac{v}{\bar w_2} + 1\Big)\, dx \\
&= \int_\Omega -\mathcal{I}_1 \cdot \nabla\Big(\log \frac{u}{\bar w_1} + 1\Big) - \mathcal{I}_1 \cdot \nabla\Big(\log \frac{v}{\bar w_2} + 1\Big)\, dx \\
&= -\int_\Omega \Big(\mathcal{I}_1 \cdot \frac{\nabla u}{u} + \mathcal{I}_2 \cdot \frac{\nabla v}{v}\Big)\, dx \\
&= -\int_\Omega \Big(d_1 u^{-1} |\nabla u|^2 + \vartheta_1 \nabla\phi \cdot \nabla u + g_{11} |\nabla u|^2 + g_{12} \nabla v \cdot \nabla u \\
&\qquad\quad d_2 v^{-1} |\nabla v|^2 + \vartheta_2 \nabla\phi \cdot \nabla v + g_{22} |\nabla v|^2 + g_{21} \nabla u \cdot \nabla v\Big)\, dx \\
&= - d_1 \int_\Omega u^{-1} |\nabla u|^2\, dx - d_2 \int_\Omega v^{-1} |\nabla v|^2\, dx \\
&\quad - \gamma_1 \vartheta_1 \int_\Omega u^2\, dx - \gamma_2 \vartheta_2 \int_\Omega v^2\, dx - (\gamma_1 \vartheta_2 + \gamma_2 \vartheta_1) \int_\Omega u\, v\, dx \\
&\quad - g_{11} \int_\Omega |\nabla u|^2\, dx - g_{22} \int_\Omega |\nabla v|^2\, dx - (g_{12} + g_{21}) \int_\Omega \nabla u \cdot \nabla v\, dx,
\end{aligned} \tag{4.11}$$

where we have used Green's first identity and $-\Delta\phi = \gamma_1 u + \gamma_2 v$ to get $\int_\Omega \nabla\phi \cdot \nabla u\, dx = \gamma_1 \int_\Omega u^2\, dx + \gamma_2 \int_\Omega u\, v\, dx$ as well as $\int_\Omega \nabla\phi\cdot\nabla v\, dx = \gamma_2 \int_\Omega v^2\, dx + \gamma_1 \int_\Omega u\, v\, dx$. On the other hand, it is readily seen that $\int_\Omega u\, v\, dx = \int_\Omega (u - \bar w_1)(v - \bar w_2)\, dx + \int_\Omega \bar w_1 \bar w_2\, dx$. In particular, we have $\int_\Omega u^2\, dx = \int_\Omega (u - \bar w_1)^2\, dx + \int_\Omega \bar w_1^2\, dx$ and $\int_\Omega v^2\, dx = \int_\Omega (v - \bar w_2)^2\, dx + \int_\Omega \bar w_2^2\, dx$. As a consequence,

$$\begin{aligned}
\partial_t H[u,v] &= - d_1 \int_\Omega u^{-1} |\nabla u|^2\, dx - d_2 \int_\Omega v^{-1} |\nabla v|^2\, dx \\
&\quad - \gamma_1 \vartheta_1 \int_\Omega (u - \bar w_1)^2\, dx - \gamma_2 \vartheta_2 \int_\Omega (v - \bar w_2)^2\, dx \\
&\quad - (\gamma_1 \vartheta_2 + \gamma_2 \vartheta_1) \int_\Omega (u - \bar w_1)(v - \bar w_2)\, dx \\
&\quad - g_{11} \int_\Omega |\nabla u|^2\, dx - g_{22} \int_\Omega |\nabla v|^2\, dx - (g_{12} + g_{21}) \int_\Omega \nabla u \cdot \nabla v\, dx - \mathcal{A}(\bar w_1, \bar w_1),
\end{aligned} \tag{4.12}$$

where $\mathcal{A}(\bar w_1, \bar w_1) = \gamma_1 \vartheta_1 \int_\Omega \bar w_1^2\, dx + \gamma_2 \vartheta_2 \int_\Omega \bar w_2^2\, dx + (\gamma_1 \vartheta_2 + \gamma_2 \vartheta_1) \int_\Omega \bar w_1 \bar w_2\, dx$.

Under the hypotheses $(\boldsymbol{\mathcal{H}1})$ and $(\boldsymbol{\mathcal{H}2})$, applying the Cauchy inequality $a\, b \le \frac{1}{2}(a^2 + b^2)$, the Poincaré inequality $\int_\Omega (u - \bar w_1)^2\, dx \le C_P \int_\Omega |\nabla u|^2\, dx$, where C_P is the Poincaré constant, and the logarithmic Sobolev inequality in Lemma 4.2, we find

$$
\begin{aligned}
\partial_t\, H[u,v] \leq & -4\, d_1\, C_L^{-1} \int_\Omega u\, \log \frac{u}{\bar{w}_1}\, dx - 4\, d_2\, C_L^{-1} \int_\Omega v\, \log \frac{v}{\bar{w}_2}\, dx \\
& - \gamma_1\, \vartheta_1 \int_\Omega (u-\bar{w}_1)^2\, dx - \gamma_2\, \vartheta_2 \int_\Omega (v-\bar{w}_2)^2\, dx \\
& - \frac{1}{2}(\gamma_1\, \vartheta_2 + \gamma_2\, \vartheta_1) \int_\Omega (u-\bar{w}_1)^2\, dx - \frac{1}{2}(\gamma_1\, \vartheta_2 + \gamma_2\, \vartheta_1) \int_\Omega (v-\bar{w}_2)^2\, dx \\
& - g_{11} \int_\Omega |\nabla u|^2\, dx - g_{22} \int_\Omega |\nabla v|^2\, dx \\
& + \frac{1}{2}(g_{12}+g_{21}) \int_\Omega |\nabla u|^2\, dx + \frac{1}{2}(g_{12}+g_{21}) \int_\Omega |\nabla v|^2\, dx - \mathcal{A}(\bar{w}_1, \bar{w}_1) \\
\leq & -4\, \min(d_1,d_2)\, C_L^{-1}\, H[u,v] - \frac{1}{2}(\gamma_2\, \vartheta_1 + \gamma_1\, (2\, \vartheta_1 + \vartheta_2)) \int_\Omega (u-\bar{w}_1)^2\, dx \\
& - \frac{1}{2}(\gamma_1\, \vartheta_2 + \gamma_2\, (2\, \vartheta_2 + \vartheta_1)) \int_\Omega (v-\bar{w}_2)^2\, dx - \mathcal{A}(\bar{w}_1, \bar{w}_1) \\
& + \frac{1}{2}(g_{12}+g_{21}-2\, g_{11}) \int_\Omega |\nabla u|^2\, dx + \frac{1}{2}(g_{12}+g_{21}-2\, g_{22}) \int_\Omega |\nabla v|^2\, dx \\
\leq & -4\, \min(d_1,d_2)\, C_L^{-1}\, H[u,v] - \frac{1}{2}(\gamma_2\, \vartheta_1 + \gamma_1\, (2\, \vartheta_1 + \vartheta_2)) \int_\Omega (u-\bar{w}_1)^2\, dx \\
& - \frac{1}{2}(\gamma_1\, \vartheta_2 + \gamma_2\, (2\, \vartheta_2 + \vartheta_1)) \int_\Omega (v-\bar{w}_2)^2\, dx - \mathcal{A}(\bar{w}_1, \bar{w}_1) + \\
& \frac{C_P^{-1}}{2}(g_{12}+g_{21}-2\, g_{11}) \int_\Omega (u-\bar{w}_1)^2\, dx + \frac{C_P^{-1}}{2}(g_{12}+g_{21}-2\, g_{22}) \int_\Omega (v-\bar{w}_2)^2\, dx \\
\leq & -4\, \min(d_1,d_2)\, C_L^{-1}\, H[u,v] - \mathcal{A}(\bar{w}_1, \bar{w}_1).
\end{aligned} \tag{4.13}
$$

In fact, $\mathcal{A}(\bar{w}_1, \bar{w}_1)$ can be rewritten as

$$
\begin{aligned}
\mathcal{A}(\bar{w}_1, \bar{w}_1) = & \gamma_1\, \vartheta_1\, \bar{w}_1^2\, |\Omega| + \gamma_2\, \vartheta_2\, \bar{w}_2^2\, |\Omega| + (\gamma_1\, \vartheta_2 + \gamma_2\, \vartheta_1)\, \bar{w}_1\, \bar{w}_2\, |\Omega| \\
= & \bar{w}_2^2\, |\Omega| \left(\gamma_1\, \vartheta_1 \left(\frac{\bar{w}_1}{\bar{w}_2}\right)^2 + (\gamma_1\, \vartheta_2 + \gamma_2\, \vartheta_1)\, \frac{\bar{w}_1}{\bar{w}_2} + \gamma_2\, \vartheta_2 \right).
\end{aligned} \tag{4.14}
$$

Due to the *compatibility condition (or the electroneutrality condition)* of the Neumann problem for ϕ, we show that $\mathcal{A}(\bar{w}_1, \bar{w}_1) = 0$. To this end, the boundary condition $\frac{\partial \phi}{\partial \nu} = 0$ on $\partial\Omega$ yields

$$
0 = \int_{\partial\Omega} \frac{\partial \phi}{\partial \nu}\, dx = \int_\Omega \nabla \cdot (\nabla \phi)\, dx = \gamma_1 \int_\Omega u\, dx + \gamma_2 \int_\Omega v\, dx, \tag{4.15}
$$

or $\gamma_1\, \bar{w}_1 = -\gamma_2\, \bar{w}_2$, which results in $\mathcal{A}(\bar{w}_1, \bar{w}_1) = 0$. This leads to the following initial value problem involving a differential inequality

$$
\begin{cases}
\partial_t\, H[u,v](t) \leq -\mathcal{B}(d_1, d_2, C_L)\, H[u,v](t), \quad t > 0, \\
H[u,v](0) = H_0,
\end{cases} \tag{4.16}
$$

where $\mathcal{B}(d_1, d_2, C_L) = 4\,\min(d_1, d_2)\,C_L^{-1}$; $H_0 = H[u, v](0)$ determined by (4.9) is a positive constant depending on $u_0(x)$ and $v_0(x)$. Gronwall's inequality yields

$$H[u, v](t) \leq e^{-\mathcal{B}t}\,H_0, \tag{4.17}$$

where $\mathcal{B} = \mathcal{B}(d_1, d_2, C_L)$. Thanks to **($\mathcal{H}$3)**, $0 < H_0 < \infty$. Since from the Csiszár-Kullback-Pinsker inequality (Lemma 4.1), the L^1-norm of $u - \bar{w}_1$ and $v - \bar{w}_2$ can be controlled by $H[u, v](t)$. The same decay rate as in (4.17) is given by

$$\|u - \bar{w}_1\|^2_{L^1(\Omega)} + \|v - \bar{w}_2\|^2_{L^1(\Omega)} \leq C_K{}^{-1}\,H_0\,e^{-\mathcal{B}t}. \tag{4.18}$$

This completes the proof of the theorem. □

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