Sections of tropicalization maps (joint work with Walter Gubler and Joe Rabinoff)

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Setting:
K field, complete with respect to a non-archimedean absolute value $|\ \ |: K \to \mathbb{R}_{\geqq 0}$.

$$\begin{array}{lcll} K^\circ & = & \{x \in K : |\, x\, | \leqq 1\} & \text{valuation ring} \\ K^{\circ\circ} & = & \{x \in K : |\, x\, | < 1\} & \text{valuation ideal} \\ \tilde{K} & = & K^\circ / K^{\circ\circ} & \text{residue field} \end{array}$$

Examples:

- $\mathbb{Q}_p$, finite extensions of $\mathbb{Q}_p, \mathbb{C}_p$
- Laurent/Puiseux series over any field
- Any field equipped with the trivial absolute value.

Setting

Y_Δ toric variety associated to the pointed fan Δ,
$T \subset Y_\Delta$ dense torus with cocharacter group N.

Kajiwara-Payne tropicalization

$$\text{trop} : Y_\Delta^{an} \to \overline{N}_{\mathbb{R}}^{\Delta},$$

where $\overline{N}_{\mathbb{R}}^{\triangle}$ is a partial compactification of the cocharacter space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ of T.

We look at a closed subscheme $\varphi : X \hookrightarrow Y_\Delta$ and its tropicalization

$$\text{Trop}_\varphi(X) = \text{image}(X^{an} \to Y_\triangle^{an} \stackrel{\text{trop}}{\to} \overline{N}_{\mathbb{R}}^{\triangle}) \subset \overline{N}_{\mathbb{R}}^{\triangle}$$

Payne and Foster,Gross,Payne: $X^{an} = \varprojlim_{\varphi} \text{Trop}_\varphi(X)$

Question

Question: Fix a tropicalization $\mathrm{Trop}_\varphi(X)$.
When does the map trop: $X^{an} \to \mathrm{Trop}_\varphi(X)$ have a continuous section $s : \mathrm{Trop}_\varphi(X) \to X^{an}$,
i.e. when is $\mathrm{Trop}_\varphi(X)$ homeomorphic to a subset of X^{an} ?

Baker-Payne-Rabinoff: If X is a curve in a torus, a continuous section exists on the locus of tropical multiplicity one.

Cueto-Häbich-W.: If $X = \mathrm{Gr}(2, n)$ is the Grassmannian of planes in n-space embedded in projective space via the Plücker embedding, then there exists a continuous section of the tropicalization map. In particular, the space of phylogenetic trees lies inside the Berkovich space $\mathrm{Gr}(2, n)^{an}$.

Our goal today

Let $\varphi : X \to Y_\Delta$ a general higher-dimensional subscheme of a toric variety.

We will give a criterion for the existence of a canonical continuous section $s : \mathrm{Trop}_\varphi(X) \to X^{\mathrm{an}}$ of the tropicalization map involving the local polyhedral structure on $\mathrm{Trop}_\varphi(X)$.

A affinoid $K-$Algebra with Banach norm $\| \quad \|$.
The Berkovich spectrum of A is the set

$$\mathcal{M}(A) = \{\text{mult. seminorms on } A, \text{bounded by } \| \quad \|\},$$

endowed with the topology of pointwise convergence.
Such $K-$affinoid spaces are the building blocks of analytic spaces.

Shilov boundary $B\big(\mathcal{M}(A)\big)$: unique minimal subset of $\mathcal{M}(A)$ on with every $f \in A$ achieves its maximum.

Analytic spaces

The $K-$affinoid space $\mathcal{M}(A)$ has a reduction Spec $\widetilde{A}$, where

$$\widetilde{A} = A^\circ / A^{\circ\circ} = \{f \in A : |f|_{\text{sup}} \leqq 1\} / \{f \in A : |f|_{\text{sup}} < 1\}.$$

Fact: Points in the Shilov boundary $B(\mathcal{M}(A)) \leftrightarrow$ minimal prime ideals in $\widetilde{A}$ (after a non-archimedean base change making A "strict").

Analytic spaces

Let X/K of finite type and X^{an} the associated Berkovich analytic space.

If $X=$ Spec A affine, then

$$X^{an} = \{\text{mult. seminorms } \gamma : A \to \mathbb{R}_{\geqq 0} \text{ extending } | \quad | \text{ on } K\}.$$

endowed with the topology of pointwise convergence.

Every $x \in X(K)$ induces a point

$$f \mapsto | f(x) |$$

in X^{an}.

Analytic spaces: example

Example: $\mathbb{G}_m = \text{Spec } K[x, x^{-1}]$.
Then $\mathbb{G}_m^{an}$ is the set of multiplicative seminorms on $K[x, x^{-1}]$ extending $|\ |$.
It is the union of the annuli $\Gamma_{r,s} = \{\gamma \in \mathbb{G}_m^{an} : s \leq \gamma(x) \leq r\}$ for $r, s > 0$. Each $\Gamma_{r,s}$ is a K-affinoid space.
In the special case $r = s$ put

$$A_r = K\langle r^{-1}x, rx^{-1}\rangle = \{ \sum_{n=-\infty}^{\infty} a_n x^n :\mid a_n \mid r^n \underset{|n|\to\infty}{\longrightarrow} 0\}$$

with Banach norm $\| \sum_{n=-\infty}^{\infty} a_n x^n \|_r = \max_n \mid a_n \mid r^n$.
Then $\Gamma_{r,r} = \mathcal{M}(A_r)$.

Now consider

$$\text{trop}: \quad \begin{array}{ccc} \mathbb{G}_m^{an} & \longrightarrow & \mathbb{R} \\ \gamma & \longmapsto & -\log \gamma(x) \end{array} .$$

For $a \in K^* \subset \mathbb{G}_m^{an}$ we have $\text{trop}(a) = -\log |a|$.
Let $w > 0$ and put $r = \exp(-w)$. The fiber $\text{trop}^{-1}(w) \subset \mathbb{G}_m^{an}$ is equal to the set of all multiplicative seminorms $\gamma : K[x, x^{-1}] \to \mathbb{R}$ extending $| \quad |$ such that $\gamma(x) = r$.
We have a natural identification

$$\text{trop}^{-1}(w) = \mathcal{M}(A_r).$$

Here the Banach norm $\| \cdot \|_r$ is multiplicative, hence an element of $\mathcal{M}(A_r)$. Therefore $B\big(\text{trop}^{-1}(r)\big) = \{\| \cdot \|_r\}$.

Tropicalization for tori

$T = \text{Spec } K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ torus

$$\begin{array}{rccc} \text{trop:} & T^{an} & \longrightarrow & \mathbb{R}^n \\ & \gamma & \longmapsto & (-\log\gamma(x_1), \ldots, -\log\gamma(x_n)) \end{array}$$

This map has natural section

$$\begin{array}{rcc} s : \mathbb{R}^n & \longrightarrow & T \\ w = (w_1, \ldots, w_n) & \longmapsto & \gamma_w \end{array}$$

with

$\gamma_w(\sum_{I=(i_1,\ldots,i_n)\in\mathbb{Z}^n} c_I x^I) = \max_I\{| c_I | \exp(-i_1 w_1 - \ldots - i_n w_n)\}$.

Then γ_w is the unique Shilov boundary point in the fiber $\text{trop}^{-1}(\{w\})$.

The image $s(\mathbb{R}^n) \subset T^{an}$ is called the skeleton of the torus. It is a deformation retract.

Tropicalization

Now let $X \hookrightarrow T$ be a closed subscheme.

Define $\mathrm{Trop}(X) = \mathrm{Image}(X^{an} \hookrightarrow T^{an} \xrightarrow{\mathrm{trop}} \mathbb{R}^n)$

Then $\mathrm{Trop}(X)$ is the closure of the image of the map

$$\begin{array}{ccccc} X(\overline{K}) & \hookrightarrow & \overline{K}^n & \longrightarrow & \mathbb{R}^n \\ x & \longmapsto & (x_1, \dots, x_n) & \longmapsto & (-\log \mid x_1 \mid_{\overline{K}}, \dots, -\log \mid x_n \mid_{\overline{K}}), \end{array}$$

if the absolute value $| \quad |$ on K is non-trivial.

Structure Theorem

If X is an irreducible subvariety of T of dimension d, the tropicalization $\mathrm{Trop}(X)$ is the support of a balanced weighted polyhedral complex Σ pure of dimension d.

Initial Degeneration

Initial degeneration: Let $X = \text{Spec } K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]/\mathfrak{a}$.
Let $w \in \text{Trop}(X)$, and choose an algebraically closed, non-trivially valued, non-archimedean field extension L/K such that there exists $t \in (L^*)^n$ with $\text{trop}(t) = w$.
Put $X_L = X \times_{\text{Spec}K} \text{Spec}L$ and take the closure of $t^{-1}X_L$ via

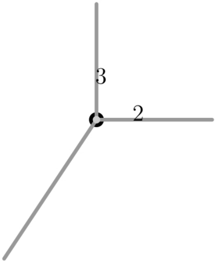
$$t^{-1}X_L \hookrightarrow T_L = \text{Spec}L[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \to \text{Spec}L^\circ[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

The special fiber of this $L^\circ-$scheme is the initial degeneration $\text{in}_w(X)$.
Tropical multiplicity:

$$\text{m}_{\text{trop}}(w) \quad = \quad \text{number of irreducible components of } \text{in}_w(X) \text{ counted with multiplicity}$$

Example: Elliptic curve

Example: $E = \{y^2 = x^3 + ax + b\}$ elliptic curve with $|\, a \,| = |\, b \,| = 1$, and $E_0 = E \cap \mathrm{Spec} K[x^{\pm 1}, y^{\pm 1}]$.
Then $\mathrm{trop}(E_0)$ looks like this:



Let $w = (1, 0)$. Then $\mathrm{in}_w(E^\circ) \simeq \mathrm{Spec}\widetilde{L}[x^{\pm 1}, y^{\pm 1}]/(y^2 - b)$ over an algebraically closed $\widetilde{L}/\widetilde{K}$.
Hence $\mathrm{m}_{\mathrm{Trop}}(w) = 2$.

Initial degeneration

Consider trop : $X^{an} \to T^{an} \to \mathbb{R}^n$. For any $w \in \text{Trop}(X)$ the preimage $\text{trop}^{-1}(w)$ is K-affinoid, hence $\text{trop}^{-1}(w) = \mathcal{M}(A_w)$.

Fact: Let $w \in \text{Trop}(X) \cap \log |K^*|^n$ for $|\ |$ non-trivial. Recall the canonical reduction $\widetilde{A}_w$ of A_w. Then there exists a finite, surjective morphism

$$\text{Spec } \widetilde{A}_w \to \text{in}_w(X).$$

over the residue field of K.

Local dimension

As above $X \hookrightarrow T$. Let $w \in \text{Trop}(X)$.
Denote by $LC_w\big(\text{Trop}(X)\big)$ the local cone of w in $\text{Trop}(X)$. Then $LC_w\big(\text{Trop}(X)\big)$ is equal to the tropicalization of $\text{in}_w(X)$ over the residue field $\widetilde{L}$ with trivial absolute value.

Definition

Define the local dimension of $\text{Trop}(X)$ at w by

$$d(w) = \dim LC_w\big(\text{Trop}(X)\big).$$

If X is equidimensional of dimension d, then $d(w) = d$ for all $w \in \text{Trop}(X)$,

Note: $d(w)$ is the dimension of the initial degeneration $\text{in}_w(X)$ over $\widetilde{L}$.

The section

Let $X \hookrightarrow T$ be a closed subscheme.

Proposition: [Gubler, Rabinoff, W.]

If $\mathrm{m}_{\mathrm{Trop}}(w) = 1$, there exists a unique irreducible component C of X of dimension $d(w)$ such that $w \in \mathrm{Trop}(C)$. Then $\mathrm{trop}^{-1}(w) \cap C^{an}$ has a unique Shilov boundary point.

Define $Z \subset \mathrm{Trop}(X)$ as the set all w such that $m_{\mathrm{Trop}}(w) = 1$.

For $w \in Z$ let C be the unique irreducible component of X of dimension $d(w)$ such that $w \in \mathrm{Trop}(C)$, and let $s(w)$ be the unique Shilov boundary point in $\mathrm{trop}^{-1}(w) \cap C^{an}$.

Section of tropicalization

Theorem 1: [Baker, Payne, Rabinoff] for (irreducible) curves and [Gubler, Rabinoff, W.] in general

Let $X \hookrightarrow T$ and $Z \subset \text{Trop}(X)$ as above. Define $s : Z \to X^{an}$ as above.
Then $\text{trop} \circ s = id_Z$ and s is continuous, i.e. s is a continuous section of the tropicalization map $\text{trop} : X^{an} \to \mathbb{R}^n$ on Z.
If Z is contained in the closure of its interior in $\text{Trop}(X)$, then s is the unique continuous section of trop on Z.

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Next step: Replace T by a toric variety.

Kajiwara-Payne tropicalization

T torus, $M = \text{Hom}(T, \mathbb{G}_m)$ character space,
$N = \text{Hom}(\mathbb{G}_m, T)$ cocharacter space. We have $M \times N \to \mathbb{Z}$.

Δ pointed rational fan in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$,

Y_Δ toric variety.

Building blocks: Let $\sigma \in \Delta$ be a cone.

$$\begin{array}{ccll} Y_\sigma & = & \text{Spec} K[\sigma^V \cap M] & \\ \cup & & & \\ O(\sigma) & = & \text{Spec} K[\sigma^\perp \cap M] & \quad \text{torus orbit.} \end{array}$$

$N_{\mathbb{R}}(\sigma) = N_{\mathbb{R}}/\langle\sigma\rangle$ cocharacter space of $O(\sigma)$.

trop: $O(\sigma)^{an} \longrightarrow N_{\mathbb{R}}(\sigma)$ torus tropicalization.

Disjoint union of all these maps:

$$\mathrm{trop} : Y_\Delta^{an} \longrightarrow \dot{\bigcup_{\sigma \in \Delta}} N_{\mathbb{R}}(\sigma) =: \overline{N}_{\mathbb{R}}^{\Delta}.$$

The right hand side carries a natural topology, such that trop is continuous.

Kajiwara-Payne tropicalization

Example:

1. $Y_\Delta = \mathbb{A}^n_K = \operatorname{Spec} K[x_1, \dots, x_n]$ affine space.

   Then $\overline{N}^\Delta_\mathbb{R} = (\mathbb{R} \cup \{\infty\})^n$ and

   $$\begin{array}{rcll} \text{trop: } (A^n_K)^{an} & \to & (\mathbb{R} \cup \{\infty\})^n & \text{is given by} \\ \gamma & \mapsto & \big(-\log\gamma(x_1), \dots, -\log\gamma(x_n)\big) & \end{array}$$

2. $Y_\Delta = \mathbb{P}^n_K$ projective space.

   Then $\overline{N}^\Delta_\mathbb{R} = \big((\mathbb{R} \cup \{\infty\})^{n+1} \backslash \{(\infty, \dots, \infty)\}\big) / \sim$

   where $(a_0, \dots, a_n) \sim (\lambda + a_0, \dots, \lambda + a_n)$ for all $\lambda \in \mathbb{R}$.

Kajiwara-Payne tropicalization

$$S(Y_\Delta) = \text{union of all skeletons in the torus orbits } O(\sigma)$$
$$\subset Y^{an}_\triangle \text{ closed subset.}$$

Let $X \hookrightarrow Y_\Delta$ closed subscheme.

$$\text{Trop}(X) = \text{image}(X^{an} \hookrightarrow Y^{an}_\Delta \xrightarrow{\text{trop}} \overline{N}^\Delta_\mathbb{R}).$$
$$= \dot{\bigcup_{\sigma\in\Delta}} \text{Trop}\big(X \cap O(\sigma)\big)$$

Let $Z \subset \text{Trop}(X)$ be the set of all w such that

$m_{\text{Trop}}(w) = 1$ (in the ambient torus orbit)

The section

Define:

$$s : Z \longrightarrow X^{an}$$

as the union of our section maps on all $Z \cap N_{\mathbb{R}}(\sigma)$.

This means: For $w \in Z \cap N_{\mathbb{R}}(\sigma)$ let C be the unique irreducible component of $X \cap O(\sigma)$ of dimension $d(w)$ such that $w \in \mathrm{Trop}(C)$ and let $s(w)$ be the unique Shilov boundary point in $\mathrm{trop}^{-1}(w) \cap C^{an}$.

Then s is a section of the tropicalization map, which is continuous on each stratum $Z \cap N_{\mathbb{R}}(\sigma)$.

The section

Question: Is s continuous on the whole space, i.e. by passing from one stratum to another?

Answer:

- Yes for irreducible curves
- Yes for the Grassmannian $\mathrm{Gr}(2, n)$
- But no in general

Example: $X = \{(x_1 - 1)x_2 + x_3 = 0\} \subset \mathbb{A}^3_K$
Trop (X) has tropical multiplicity one everywhere.

$$\mathrm{Trop}(X \cap \mathbb{G}^3_{m,K}) \supset P = \{w \in \mathbb{R}^3 : w_1 = 0, w_3 \geqq w_2\}.$$

$$\begin{array}{ccll} w_n & = & (0, n, 2n) \in P & \text{for all } n. \\ \downarrow & & & \\ w & = & 0 + \langle\sigma\rangle \in \mathbb{R}^3/\langle\sigma\rangle & \text{for } \sigma = \langle e_2, e_3\rangle \end{array}$$

Note: $X \cap O(\sigma) = O(\sigma)$.

The section

Since $(x_1 - 1)x_2 + x_3 = 0$, we have

$s(w_n)(x_1 - 1) = s(w_n)(-x_3/x_2) = e^{-n} \underset{n\to\infty}{\longrightarrow} 0.$

But the Shilov boundary norm in $0 \in \text{Trop}\,\big(X \cap O(\sigma)\big) = \mathbb{R}$

maps $(x_1 - 1)$ to 1 !

Problem: $\pi_\sigma : \mathbb{R}^3 \longrightarrow \mathbb{R}^3/ < \sigma >$ maps P to a point, whereas $X \cap O(\sigma) = O(\sigma) \simeq \mathbb{G}_{m,K}$. Hence for $w = 0+ < \sigma > \in \mathbb{R}^3/ < \sigma >$ we find $d(w) = 1$ which does not match the dimension of $\pi_\sigma(P)$.

Lemma [Osserman, Rabinoff]

$P \subset N_{\mathbb{R}}$ polyhedron with closure $\overline{P}$ in $\overline{N}_{\mathbb{R}}^{\Delta}$.
$\sigma \in \Delta, \pi_{\sigma} : N_{\mathbb{R}} \to N_{\mathbb{R}} / < \sigma >= N_{\mathbb{R}}(\sigma)$ the projection map.

i) $\overline{P} \cap N_{\mathbb{R}}(\sigma) \neq \emptyset$ if and only if the recession cone of P meets the relative interior of σ.
ii) If $\overline{P} \cap N_{\mathbb{R}}(\sigma) \neq \emptyset$, then $\overline{P} \cap N_{\mathbb{R}}(\sigma) = \pi_{\sigma}(P)$.

Theorem [Guber, Rabinoff, W.]

Let $X \hookrightarrow Y_\Delta$ be a closed subscheme. Let $(w_n)_{n\geq 1}$ be a sequence in $\mathrm{Trop}(X) \cap N_{\mathbb{R}}$ with $\mathrm{m}_{\mathrm{trop}}(w_n) = 1$, converging to $w \in \mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$ with $\mathrm{m}_{\mathrm{trop}}(w) = 1$.

Suppose that there exists a polyhedron $P \subset \mathrm{Trop}(X) \cap N_{\mathbb{R}}$ containing all w_n with $\dim P = d(w_n)$ for all n.
If $\dim \pi_\sigma(P) = d(w)$, then

$$s(w_n) \underset{n\to\infty}{\longrightarrow} s(w).$$

Idea of proof: Reduce to the case $X \cap T$ irreducible and dense in X. Construct a toric morphism $Y_\Delta \to Y_{\Delta'}$ to a toric variety $Y_{\Delta'}$ of dimension $d = \dim(X)$, such that

$$\mathrm{Trop}(X) \subset \overline{N}_\Delta \to \overline{N'}_{\Delta'}$$

is injective on P and $\pi_\sigma(P)$. Then $s(w_n)$ is mapped to the skeleton $S(Y'_{\Delta'})$ via $X^{\mathrm{an}} \to Y_\Delta^{an} \to (Y'_{\Delta'})^{an}$. The preimage of the skeleton $S(Y'_{\Delta'})$ in X^{an} is closed.
Let ξ be an accumulation point of $(s(w_n))_n$, then ξ lies in the preimage of $S(Y'_{\Delta'})$ and in $\mathrm{trop}^{-1}(w)$. We show that this intersection only contains Shilov boundary points, hence $\xi = s(w)$.

The section

Corollary 1:

$X \hookrightarrow Y_\Delta$ as above with $X \cap T$ dense in T and all $X \cap O(\sigma)$ equidimensional of dimension $d(\sigma)$. Put $d = d(0) = \dim X \cap T$.

If $\mathrm{Trop}(X) \cap N_\mathbb{R}$ can be covered by finitely many $d-$dimensional polyhedra P such that $\dim \pi_\sigma(P) = d_\sigma$ whenever $\overline{P} \cap N_\mathbb{R}(\sigma) \neq \emptyset$,, then

$$s : \{w \in \ \mathrm{Trop}(X) : \mathrm{m_{trop}}(w) = 1\} \longrightarrow X^{an}$$

is continuous.

This explains the existence of a continuous section for $X = Gr(2, n)$, where $\mathrm{m_{Trop}} = 1$ everywhere. However, the combinatorial techniques from [Cueto, Häbich, W.] are still necessary to show that the prerequisites of Corollary 1 are fulfilled.

The section

Corollary 2:

If $X \cap T$ is dense in X and for all $\sigma \in \Delta$ either $X \cap O(\sigma) = \emptyset$ or X intersects $O(\sigma)$ properly (i.e. in dimension $d - \dim \sigma$), then

$$s : \{w \in \text{Trop}(x) : \mathrm{m}_{\text{trop}}(w) = 1\} \longrightarrow X^{an}$$

is continuous.

Example: Tropical compactifications.

Tropical skeleton

More generally: For $X \hookrightarrow Y_\triangle$ define the tropical skeleton

$$S_{\mathrm{Trop}}(X) = \{\zeta \in X^{an} : \zeta \text{ is a Shilov boundary point in } \mathrm{trop}^{-1}(w) \text{ for } w = \mathrm{trop}\ \zeta\}.$$

The previous results can be generalized to criteria for $S_{\mathrm{Trop}}(X)$ to be closed.