# Buildings and Berkovich Spaces

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This talk reports on joint work with Amaury Thuillier and Bertrand Rémy (Lyon). Our results generalize results of Vladimir Berkovich who investigated the case of split groups.

# Non-archimedean fields

$K$ non-Archimedean field, i.e. $K$ is complete with respect to a non-trivial absolute value $| \ |_K$ satisfying

$$|a+b|_K \leqq \max\{|a|_K, |b|_K\}.$$

$K$ is called discrete if the value group $|K^*| \subset \mathbb{R}^*$ is discrete.

Non-archimedean analysis has special charms:

$$\sum_{n=1}^{\infty} a_n \text{ converges if and only if } a_n \to 0.$$

# Non-archimedean fields

Examples:

- $K = k((T))$ formal Laurent series over any ground field $k$ with $|\sum_{n \geq n_0} a_n T^n| = e^{-n_0}$ if $a_{n_0} \neq 0$
- $K = \mathbb{C}\{\{T\}\}$ Puiseux series
- $K = \mathbb{Q}_p$, the completion of $\mathbb{Q}$ with respect to $|x| = p^{-v_p(x)}$
- algebraic extensions of $\mathbb{Q}_p$
- $K = \mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$

# Semisimple groups

$G$ semisimple group over $K$, i.e.

$G \hookrightarrow GL_{n,K}$ closed algebraic subgroup such that

$\mathrm{rad}(G)(=$ biggest connected solvable normal subgroup$) = 1$

Examples: $SL_n, PGL_n, Sp_{2n}, SO_n$ over $K$
$SL_n(D)$ $D$ central division algebra over $K$

# Main result

Goal: Embed the Bruhat-Tits building $\mathfrak{B}(G, K)$ associated to $G$ in the Berkovich analytic space $G^{an}$ associated to $G$.

Hope: Investigate the building with the help of the ambiant Berkovich space $G^{an}$.

# Bruhat-Tits buildings: Example

Archimedean Example:

$G = SL(2, \mathbb{R})$
$H = SO(2, \mathbb{R})$ maximal compact subgroup
$G/H = \mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$
upper half-plane

Non-Archimedean analog:

$p$ prime number
$G = SL(2, \mathbb{Q}_p)$
$H = SL(2, \mathbb{Z}_p)$ maximal compact subgroup.
$G/H$ is a totally disconnected topological space.

Note: $\mathbb{H} = \{\text{norms on } \mathbb{R}^2\}/\text{scaling}.$

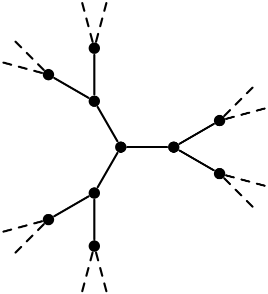
Goldman-Iwahori:

$\mathfrak{B}(SL_2, \mathbb{Q}_p) = \{\text{Non-archimedean norms on } \mathbb{Q}_p^2\}/ \text{ scaling}$

- Topology of pointwise convergence
- $SL(2, \mathbb{Q}_p)-$action
- Stabilizer of the norm $\binom{a}{b} \mapsto \max\{|a|, |b|\}$ is the maximal compact subgroup $SL(2, \mathbb{Z}_p)$

# Bruhat-Tits buildings: Example

- $\mathfrak{B}(SL_2, \mathbb{Q}_p)$ is an infinite $(p+1)-$valent tree:



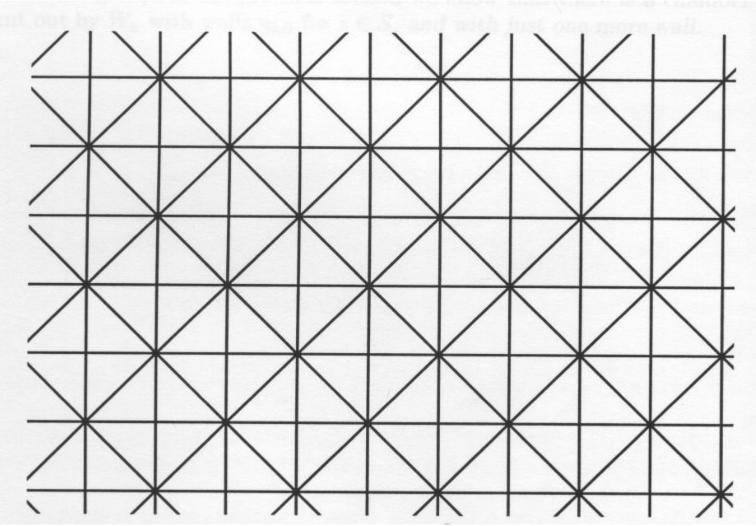
$(p = 2)$.

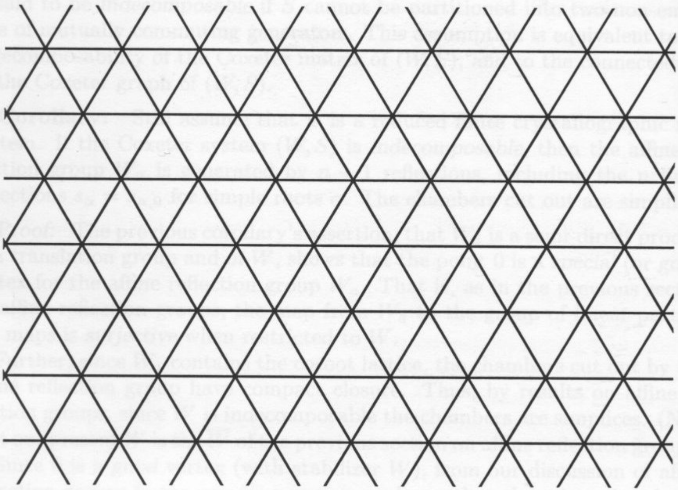
In general: The building $\mathfrak{B}(G, K)$ is obtained by glueing real vector spaces (apartments).
Every maximal split torus $T \subset G$, i.e. $T \simeq \mathbb{G}_{m,K}^r$, induces an apartment $A(T)$, which is defined as the real cocharacter space $A(T) = \mathrm{Hom}_K(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{R}$.
The glueing process is defined with deep (and quite technical) results by Bruhat and Tits.

$\mathfrak{B}(G, K)$ is a complete metric space with a continuous $G(K)-$action.

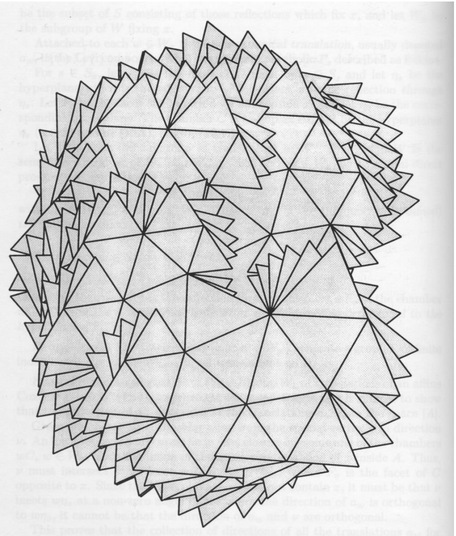
If $K$ is discrete, $\mathfrak{B}(G, K)$ carries a (poly-)simplicial structure.

# Apartment for $Sp_4$





# Some part of $\mathcal{B}(PGL_3, \mathbb{Q}_p)$



P. Garrett: Buildings and classical groups

# Why are Bruhat-Tits buildings useful?

- $\mathfrak{B}(G, K)$ is a nice space on which $G(K)$ acts
- Cohomology of arithmetic groups (Borel-Serre)
- $\mathfrak{B}(G, K)$ encodes information about the compact subgroups of $G(K)$
- Representation theory of $G(K)$ (Schneider-Stuhler)
- Bruhat-Tits buildings are non-Archimedean analogs of Riemann symmetric spaces of non-compact type
- Buildings can be used to prove results for symmetric spaces (e.g. Kleiner-Leeb)

# Berkovich Spaces

A Berkovich space is a non-Archimedean analytic space with good topological properties.

Archimedean case:
$X$ smooth projective variety over $\mathbb{C}$. Then $X(\mathbb{C})$ is a complex projective manifold.

Non-archimedean case:
$X$ smooth projective variety over $K$. Then $X(K)$ inherits a non-Archimedean topology from $K$ with bad topological properties, e.g. it is totally disconnected.

# Berkovich Spaces

**Tate, Raynaud...** Define non-Archimedean analytic functions by a suitable Grothendieck topology

**Berkovich** Enlarge $X(K)$ to a topological space $X^{an}$ with good properties.

# Berkovich unit disc

Example: The Berkovich unit disc

Assume for simplicity that $K$ is algebraically closed.

$A = K\{z\} = \{$ formal series $f(z) = \sum_{n \geq 0} a_n z^n$ with $a_n \to 0\}$

$\| f \| = \max_n |a_n|_K$ Gauss norm on $A$

# Berkovich unit disc

$\mathcal{M}(A) = \{$ bounded multiplicative seminorms on $A$ extending $| \ |_K \}$ is the Berkovich unit disc.

Hence every $\gamma \in \mathcal{M}(A)$ is a function $\gamma : A \to \mathbb{R}_{\geqq 0}$ satisfying

- $\gamma|_K = | \ \ |_K$
- $\gamma(fg) = \gamma(f)\, \gamma(g)$
- $\gamma(f + g) \leqq \gamma(f) + \gamma(g)$
- $\gamma \leq c \, || \ ||$

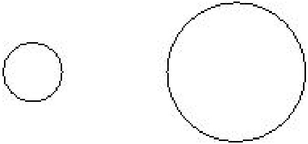
Every $a \in K$ with $|a|_K \leqq 1$ induces a point $|f|_a = |f(a)|_K$ in $\mathcal{M}(A)$

The Gauss norm is multiplicative, i.e. a point in $\mathcal{M}(A)$.

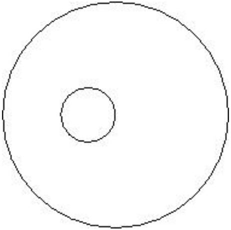
# Non-archimedean balls

The other seminorms in $\mathcal{M}(A)$ can be described with closed non-Archimedean discs $D(a,r) = \{x \in K : |x-a| \leqq r\}$

Note: Two non-Archimedean closed discs are either disjoint



or nested



# Berkovich unit disc

Basic fact: The Gauss norm is the supremum norm on $D(0,1)$.

The Berkovich unit disc consists of the following points:

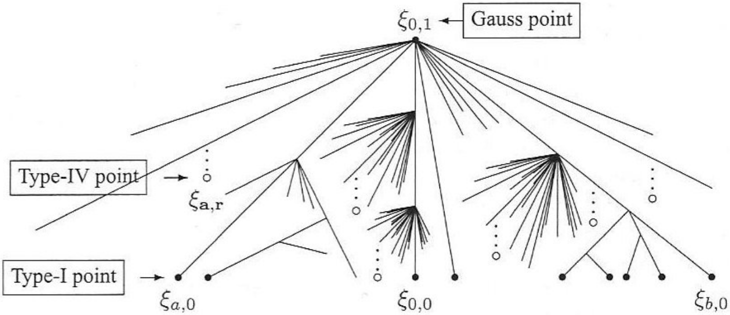
Points of type 1: $|f|_a = |f(a)|_K$ for $a \in D(0,1)$.

Points of type 2: $|f|_{a,r} = \sup_{x \in D(a,r)} |f(x)|_K$ for $D(a,r) \subset D(0,1)$ and $r \in |K^*|$

Points of type 3: $|f|_{a,r} = \sup_{x \in D(a,r)} |f(x)|$ for $D(a,r) \subset D(0,1)$ and $r \notin |K^*|$

Points of type 4: $|f|_{\underline{a},\underline{r}} = \lim_{n \to \infty} |f|_{a_n,r_n}$ for a nested sequence $D(a_1,r_1) \supset D(a_2,r_2) \ldots$ of closed discs in $D(0,1)$

# Berkovich unit disc




(from J.H. Silverman: The arithmetic of dynamical systems)

# Berkovich unit disc

Endow the Berkovich unit disc $\mathcal{M}(A)$ with the topology of pointwise convergence of seminorms evaluated on $A$.

Then $\mathcal{M}(A)$ is a compact, uniquely path-connected Hausdorff space containing $\{x \in K : |x|_K \leqq 1\}$ as a dense subspace.

# Berkovich spaces

Similary one can define Berkovich discs of any radius $r > 0$.

Berkovich affine line:

$$(\mathbb{A}^1_K)^{an} = \text{union of all Berkovich discs of positive radius}$$
$$= \{\text{multiplicative seminorms on } K[z]\}.$$

Berkovich projective line:

$(\mathbb{P}^1_K)^{an}$ can be constructed by glueing two Berkovich unit discs.

# Analytification of algebraic varieties

In general:

$X = \text{Spec } A$ for $A = K[x_1, \ldots, x_n]/\mathfrak{a}$

Berkovich space $X^{an}$ corresponding to $X$ :

$X^{an} = \{\text{multiplicative seminorms on } A \text{ extending } |\ |_K\}$

An analogous definition over the complex numbers yields $X(\mathbb{C})$ by a theorem of Gelfand-Mazur.

# Berkovich spaces

Berkovich spaces have found a variety of applications, e.g.

- to prove a conjecture of Deligne on vanishing cycles (Berkovich)
- in local Langlands theory (Harris-Taylor)
- to develop a $p-$adic avatar of Grothendieck's "dessins d'enfants" (André)
- to develop a $p-$adic integration theory over genuine paths (Berkovich)
- for $p-$adic harmonic analysis and $p-$adic dynamics with applications in Arakelov Theory (Baker, Chambert-Loir, Rumely, Thuillier,...)

# Embedding Theorem

$G$ semisimple algebraic group over $K$

$G^{an}$ Berkovich space associated to $G$

We define a continuous, $G(K)-$ equivariant embedding

$$\nu : \mathfrak{B}(G, K) \longrightarrow G^{an}$$

using the following theorem:

# Embedding Theorem

**Theorem**

i) For all $x \in \mathfrak{B}(G, K)$ there exists an (affinoid) subgroup $G_x = \mathcal{M}(A_x) \subset G^{an}$ such that

$$G_x(L) = \mathrm{Stab}_{G(L)}(x)$$

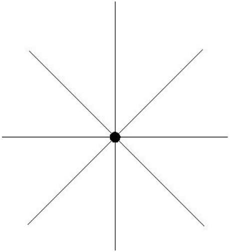
for all non-Archimedean fields $L \supset K$.

ii) $G_x$ has a unique maximal point in $G^{an}$ (Shilov boundary point), i.e. there exists a unique $\nu(x) \in G^{an}$ such that all $f \in A_x$ achieve their maximum on $\nu(x)$.

# Embedding Theorem

Tools: Bruhat-Tits theory, Berkovich's characterization of Shilov boundary points, descent theory for affinoids

New idea: Any point $x$ becomes special



after base extension with a suitable $L/K$.

# Embedding Theorem: Example

Example $G = SL_2$
$T \subset G$ torus of diagonal matrices

$$A(T) = \mathrm{Hom}(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}\mu \text{ for } \mu : a \mapsto \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

$$U_- = \left\{ \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} : u \in K \right\} \qquad U_+ = \left\{ \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} : v \in K \right\}$$

$\Omega = U_- T U_+ \subset SL_2$ big cell

Note that $\Omega = \mathrm{Spec}\ K[a, a^{-1}, u, v]$.

# Embedding Theorem: Example

The embedding $\nu$ is constructed apartment-wise. It maps $A(T)$ to the analytified big cell $\Omega^{an} \subset SL_2^{an}$.

Explicit description: Let $x\mu \in A(T)$.

Then $\nu(x\mu) \in \Omega^{an}$ is the following multiplicative seminorm on $K[a, a^{-1}, u, v]$:

$$| \sum_{\substack{k \in \mathbb{Z} \\ m,n \in \mathbb{N}_0}} c_{kmn} \; a^k u^m v^n |_{\nu(x\mu)} = \max_{k,m,n} |c_{kmn}|_K |e^{x(m-n)}.$$

In particular, for $0 \in A(T)$ we get

$$| \sum_{k,m,n} c_{kmn} a^k u^m v^n |_{\nu(0)} = \max_{k,m,n} |c_{kmn}|.$$

# Compactifications

Application: Compactifications of Bruhat-Tits buildings

$G$ semisimple algebraic group over $K$

$P \subset G$ parabolic subgroup

$G/P$ proper $K-$ variety

Example: $G = SL_n$ over $K$

$F = (V_0 \subset \ldots \subset V_k)$ flag of linear subspaces of $K^n$

$P = \mathrm{Stab}(F) \subset SL_n$

$G/P$ flag variety.

**Definition** $\nu_P : \mathfrak{B}(G,K) \xrightarrow{\nu} G^{an} \to (G/P)^{an}$

The closure of the image of $\mathfrak{B}(G,K)$ under $\nu_P$ is a compactification $\overline{\mathfrak{B}_P(G,K)}$ of $\mathfrak{B}(G,K)$ (or of some almost simple factors).

**Theorem** $$\overline{\mathfrak{B}}_P(G,K) = \bigcup_{Q \text{ "good" parabolic}} \mathfrak{B}(Q_{ss},K)$$

**Theorem** Any two points $x,y$ in $\overline{\mathfrak{B}}_P(G,K)$ are contained in one compactified apartment.

**Theorem** (Mixed Bruhat decomposition)
Let $x,y \in \overline{\mathfrak{B}}_P(G,K)$ with stabilizers $P_x, P_y \subset G(K)$.

Then $G(K) = P_x N(K) P_y$.

# Compactifications

Example:

$$G = SL_n \text{ over } K.$$

$$P = \left\{ \begin{pmatrix} * & \cdots & * & * \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & * \end{pmatrix} \right\} \text{ the stabilizer of a hyperplane}$$

$$\mathfrak{B}(G, K) = \{\text{non-Archimedean norms on } K^n\}/\text{scaling}$$

$$\cap$$

$$\overline{\mathfrak{B}}_P(G, K) = \{\text{non-Archimedean seminorms on } K^n\}/\text{scaling}$$

$$\cap$$

$$(G/P)^{an} = (\mathbb{P}^{n-1})^{an}$$